

Name:

MWF 9 Oliver Knill
MWF 9 Chao Li
MWF 10 Gijs Heuts
MWF 10 Yu-Wen Hsu
MWF 10 Yong-Suk Moon
MWF 11 Rosalie Belanger-Rioux
MWF 11 Gijs Heuts
MWF 11 Siu-Cheong Lau
MWF 12 Erick Knight
MWF 12 Kate Penner
TTH 10 Peter Smillie
TTH 10 Jeff Kuan
TTH 10 Yi Xie
TTH 11:30 Jeff Kuan
TTH 11:30 Jameel Al-Aidroos

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

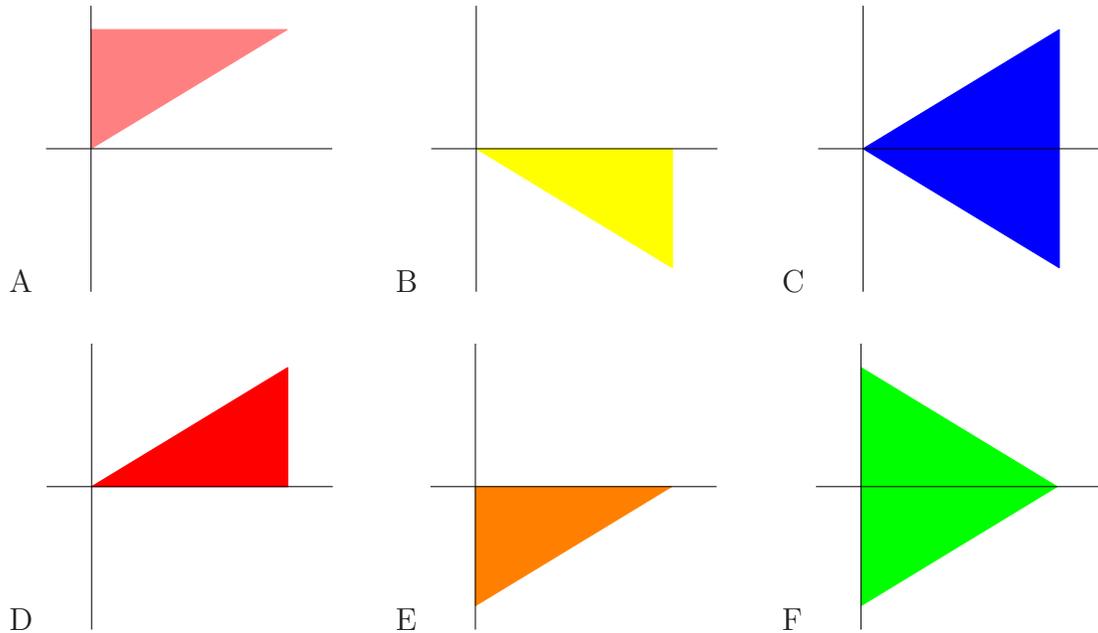
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points), no justifications needed

- 1)  T  F The length of the gradient  $\nabla f(0, 0)$  is the maximal directional derivative  $|D_{\vec{v}}f(0, 0)|$  among all unit vectors  $\vec{v}$ .
- 2)  T  F The relation  $f_{xxyyxx} = f_{xyxyyx}$  holds everywhere for  $f(x, y) = \cos(\exp(x^{10}) + \sin(x - y))$ .
- 3)  T  F  $\int_0^4 \int_0^{4x} f(x, y) dydx = \int_0^{16} \int_{y/4}^{16} f(x, y) dx dy$ .
- 4)  T  F  $g(x, y) = \int_y^0 \int_0^x f(s, t) ds dt$  satisfies  $g_{xy} = -f(x, y)$ .
- 5)  T  F If  $\vec{r}(u, v)$  is a parametrization of the level surface  $f(x, y, z) = c$ , then  $\nabla f(\vec{r}(u, v)) \cdot \vec{r}_v(u, v) = 0$ .
- 6)  T  F If  $D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f(a, b) = 3$  and  $D_{\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle} f(a, b) = 5$ , then  $D_{\vec{v}}f(a, b) \geq 0$  for all unit directions  $\vec{v}$ .
- 7)  T  F Given a parametrization  $\vec{r}(t)$  of a curve and a function  $f(x, y)$  we have  $\frac{d}{dt} f(\vec{r}(2t)) = 2\nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$  at  $t = 0$ .
- 8)  T  F If  $u(t, x)$  solves both the heat and wave equation, then  $u_t = c u_{tt}$  for some constant  $c$ .
- 9)  T  F If the Lagrange multiplier  $\lambda$  at a solution to a Lagrange problem is negative then this point is neither a maximum nor a minimum.
- 10)  T  F The equation  $f_x^2 + f_y^2 + f_z^2 = 1$  is an example of a partial differential equation.
- 11)  T  F If the discriminant  $D$  of  $f(x, y)$  is zero at  $(0, 0)$  then  $\nabla f(0, 0) = \langle 0, 0 \rangle$ .
- 12)  T  F If  $f(x, y, z) = 0$  describes the unit sphere, then the gradient  $\nabla f$  points outwards.
- 13)  T  F If  $f(x, y)$  is a continuous function then  $\int_0^2 \int_0^1 f(x, y) dx dy = \int_0^2 \int_0^1 f(y, x) dx dy$ .
- 14)  T  F The point  $(5, 5, 5)$  is a critical point of  $f(x, y, z) = x + y + z$ .
- 15)  T  F Assume  $\nabla f(0, 0) = \langle 0, 0 \rangle$  with discriminant  $D > 0$ , then  $-f(x, y)$  has the same critical point  $(0, 0)$  with discriminant  $D < 0$ .
- 16)  T  F  $\int \int_R |\nabla f|^2 dx dy$  is the surface area of the cubic paraboloid  $z = f(x, y) = x^3 + y^3$  defined over the region  $R$ .
- 17)  T  F If  $D(x, y)$  is the discriminant of  $f$  at  $(x, y)$  then the following poetic formula of the directional derivative of the discriminant holds:  $D_{\langle 1, 0 \rangle} D = \partial_x D$ .
- 18)  T  F Assume  $f(x, y) = -x^2 + y^4$  and a curve  $\vec{r}(t)$  satisfying  $\vec{r}'(t) = \nabla f(\vec{r}(t))$ , then  $\frac{d}{dt} f(\vec{r}(t)) \geq 0$  for all  $t$ .
- 19)  T  F The Lagrange equations for extremizing  $f(x, y)$  under the constraint  $g(x, y) = c$  have the same solutions as the Lagrange equations for extremizing  $F = f + g$  under the constraint  $g = c$ .
- 20)  T  F If  $f$  is a maximum under the constraint  $g = 1$  at  $(0, 0)$ , and  $(0, 0)$  is not a critical point for both  $f$  and  $g$ , then the level curves of  $f$  and  $g$  have the same tangent line at  $(0, 0)$ .

Problem 2) (10 points) No justifications needed

a) (6 points) Match the regions with the integrals. Each integral matches one region A–F.



Enter A-F	Integral
	$\int_{-1}^1 \int_{ y }^1 f(x, y) \, dx dy$
	$\int_0^1 \int_0^x f(x, y) \, dy dx$
	$\int_{-1}^0 \int_{-y}^1 f(x, y) \, dx dy$
	$\int_0^1 \int_{x-1}^0 f(x, y) \, dy dx$
	$\int_0^1 \int_{x-1}^{1-x} f(x, y) \, dy dx$
	$\int_0^1 \int_x^1 f(x, y) \, dy dx$

b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

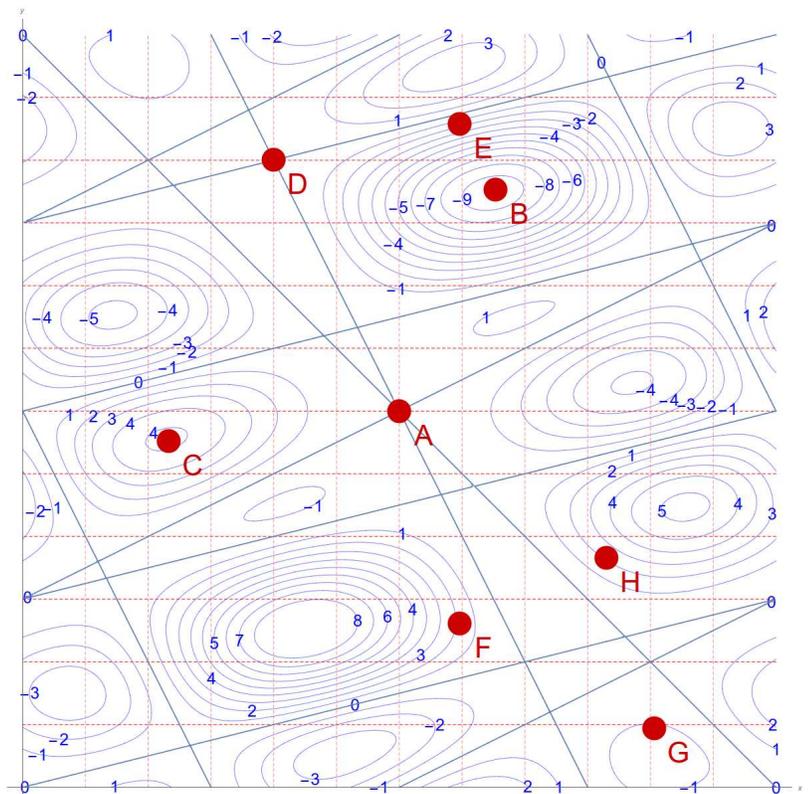
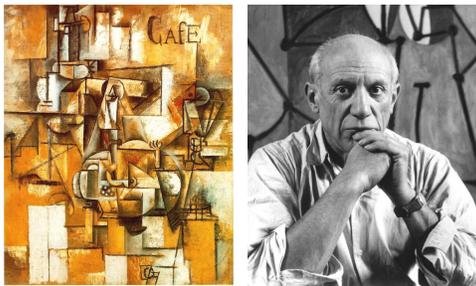
Fill in 1-4	Name
	Laplace
	Wave
	Transport
	Heat

Equation number	Formula for PDE
1	$\frac{\partial}{\partial t} u - \frac{\partial}{\partial y} u = 0$
2	$\frac{\partial}{\partial t} u - \frac{\partial^2}{\partial x^2} u = 0$
3	$\frac{\partial^2}{\partial t^2} u - \frac{\partial^2}{\partial x^2} u = 0$
4	$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$

Problem 3) (10 points)

a) (7 points) The following contour map is inspired by a cubistic style of Picasso. Each of the points A-H fit exactly once

A point where $f_x \neq 0, f_y = 0$	
A saddle point of $f$	
A local maximum of $f$	
A critical point with $D = 0$	
A local minimum	
A point with $f_y \neq 0, f_x = 0$	
$ \nabla f $ maximal among A-H	
Point where $D_{(-1,1)/\sqrt{2}}f = 0$	



The painting "Pigeon with Green Peas" by Pablo Picasso was stolen in 2010. The thief got scared and disposed it to trash shortly after the theft. The garbage was emptied and taken away, the painting lost for ever. Or the thief had been clever ...

b) (3 points) Given a function  $f(x, y)$  and a curve  $\vec{r}(t)$ . Let  $L$  be the linearization of  $f$  at  $\vec{r}(0)$ . Each of the following 3 vectors  $\vec{a}, \vec{b}, \vec{c}$  is placed exactly twice in the puzzle to the right below.

$\vec{a} = \nabla f(\vec{r}(0))$
$\vec{b} = \vec{r}'(0)$
$\vec{c} = \vec{r}(t) - \vec{r}(0)$

$\frac{d}{dt}f(\vec{r}(t))_{t=0} =$	<input type="text"/>	$\cdot$	<input type="text"/>	
$L(\vec{r}(t)) =$	$f(\vec{r}(0)) +$	<input type="text"/>	$\cdot$	<input type="text"/>
<input type="text"/>	$=$	$\lim_{t \rightarrow 0} \frac{1}{t}$	<input type="text"/>	

Problem 4) (10 points)

a) (5 points) Find the tangent plane to the **skate board ramp**

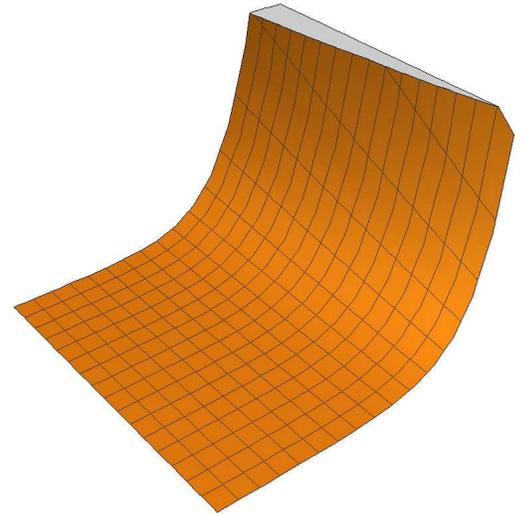
$$z - f(x, y) = z - \sqrt{x^{30}y^3 + x} = 0$$

at the point  $(1, 2, 3)$ .

b) (5 points) Estimate

$$f(1.006, 1.98) = \sqrt{1.006^{30} \cdot 1.98^3 + 1.006}$$

by linearizing the function  $f(x, y)$  at  $(1, 2)$ .



Problem 5) (10 points)

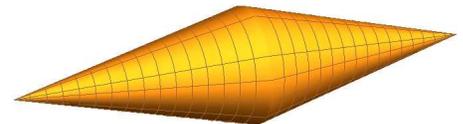
A **croissant** of length  $2h$  and radius  $r$  in the shape of two cones has fixed volume

$$V(r, h) = \frac{2\pi r^2 h}{3} = 18.$$

Use Lagrange to find the values  $r$  and  $h$  for which the surface area

$$A(r, h) = 2\pi r \sqrt{r^2 + h^2}$$

is minimal. **Hint:** as you have seen in homework, it is much more convenient to minimize  $f(r, h) = A(r, h)^2$  instead.

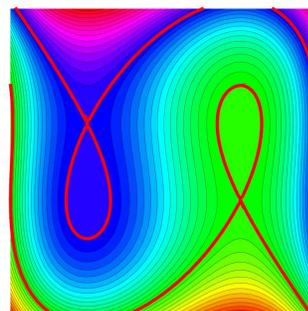


Problem 6) (10 points)

Find the local maxima, minima and saddle points of the **tadpole function**

$$f(x, y) = 3y^2 + 4x^3 + 2y^3 - 12x.$$

*First tadpole:* “What is your favorite book?” *Second tadpole:* “Metamorphosis by Kafka. What is your favorite year?” *First tadpole:* “Leap year!” *Both croak with laughter. Then, sick of frog jokes, they turn green.*



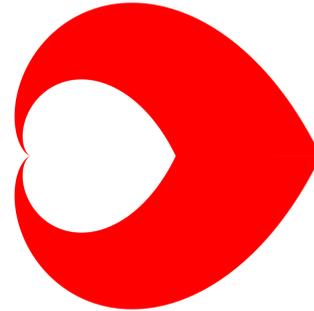
Problem 7) (10 points)

Find the area of the heart shaped polar region

$$(\theta - \pi)^2 \leq r \leq 2(\theta - \pi)^2$$

with  $0 \leq \theta \leq 2\pi$ .

*Warning: Valentine cards displaying  
“You are my  $r < (\theta - \pi)^2$ !” do not always work.*



Problem 8) (10 points)

Mathematica 10 does not give an elementary expression for the integral

$$\int_0^1 \int_{\exp(y)}^e \frac{1}{\log(x)} dx dy ,$$

where  $\log$  is the natural log. You can! “Humans are awesome 2014”! <https://www.youtube.com/watch?v=ZBCOMG2F2Zk>

The logarithmic integral  $\text{Li}(x) = \int_0^x dt/\log(t)$  is important in number theory. It was Gauss who proposed first that the number  $\pi(x)$  of primes smaller than  $x$  is about  $\text{Li}(x)$ . It is now known that  $0.89 \text{Li}(x) \leq \pi(x) \leq 1.11 \text{Li}(x)$  for all large enough  $x$ . P.S. Mathematica can solve the double integral of course, but only if told to “FullSimplify”.



Problem 9) (10 points)

Compute the weighted surface area

$$\int \int_R (u^2 + v^2) |\vec{r}_u \times \vec{r}_v| dudv$$

of the monkey saddle parametrized by  $\vec{r}(u, v) = \langle u, v, u^3 - 3uv^2 \rangle$  over the domain  $R : u^2 + v^2 \leq 1$ . This quantity is also known as the moment of inertia of the surface. Spin that monkey!

