Math 21a Fall 13 Review II

Grad | Curl | Div

Ampere

Stokes

Gauss

Oliver Knill

December 11, 2013

What can we match?

- Parametric surfaces
- Quadrics
- Vector fields
- integration regions
- Solids
- Contour maps
- PDE's
- Critical points
- Contour surfaces
- Objects in general
- Derivatives
- Integrals
- Curves in the plane
- Curves in Space
- Line integrals

Vector Calculus

Objects

- Curves
- Surfaces
- Solids

Fields

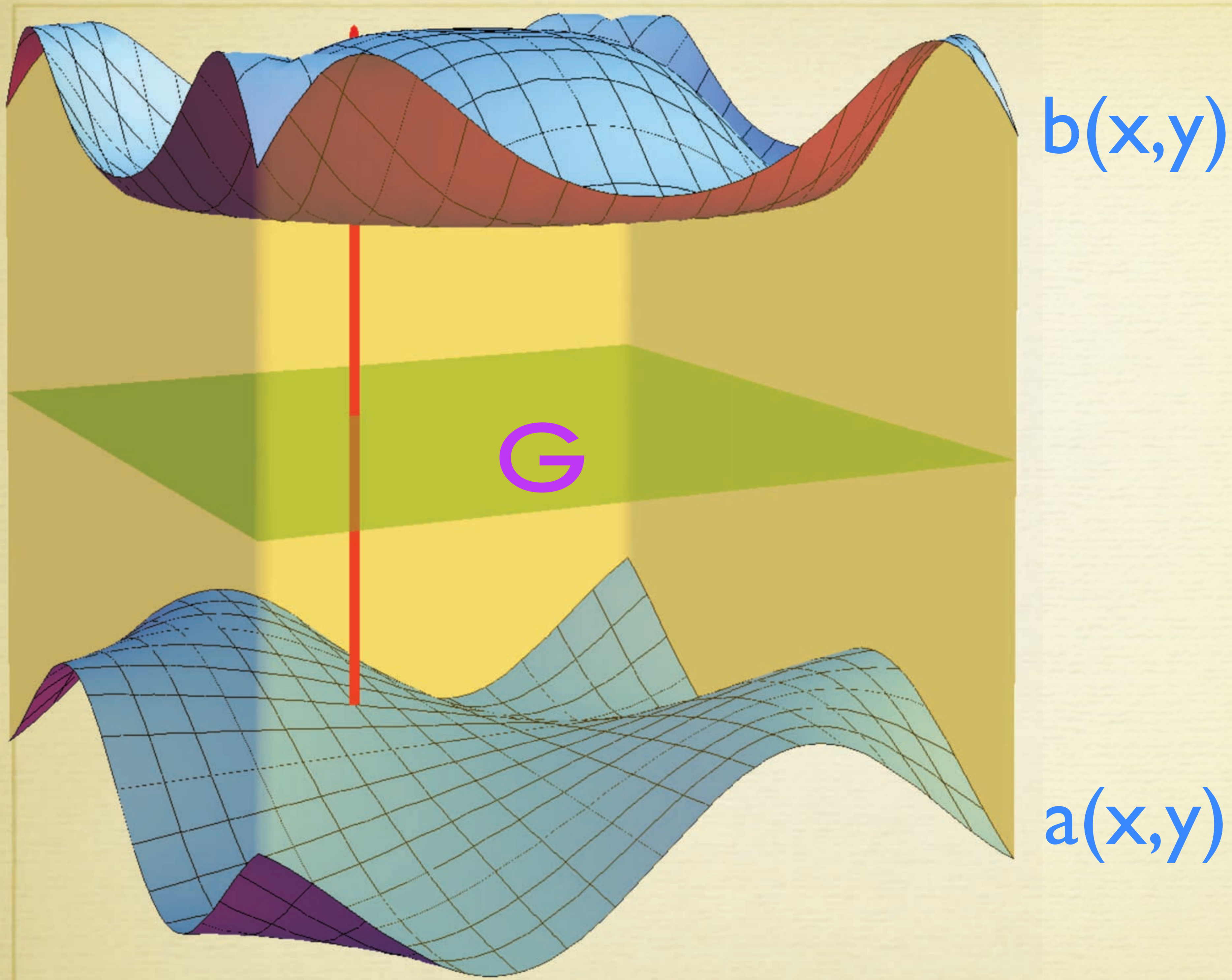
- Vector Fields
- Vector Fields
- Scalar Fields

Integrals

- Line integrals
- Flux integrals
- Triple integrals

Derivatives

- Grad
- Curl
- Div



$$\iint_G \int_a^b f(x,y,z)\ dz\ dx\ dy$$

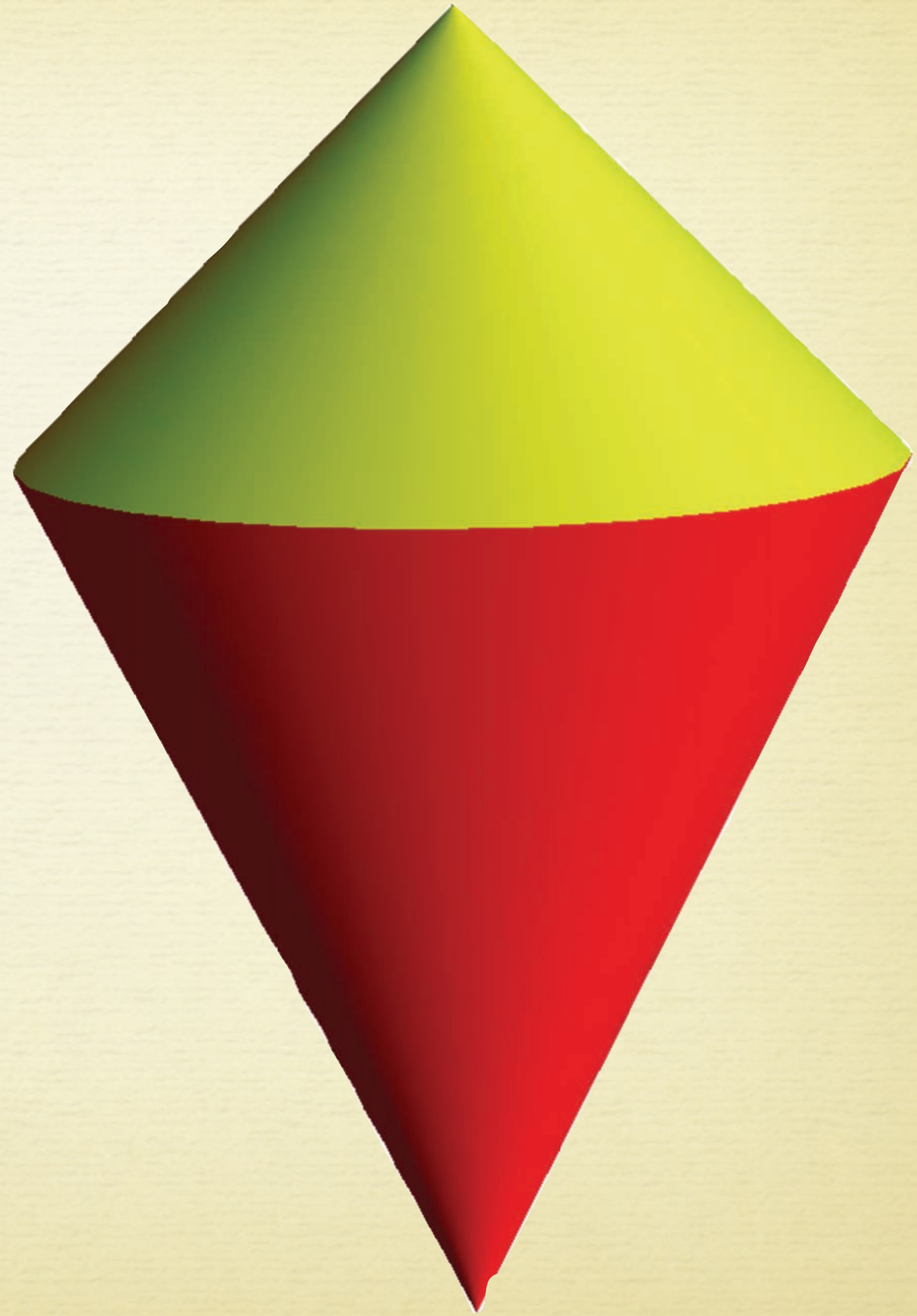
Problem

Integrate $\sqrt{x^2 + y^2}$

over the solid

$z^2 < 1 - x^2 - y^2$

$z^2 > 2x^2 - 2y^2 - 2$



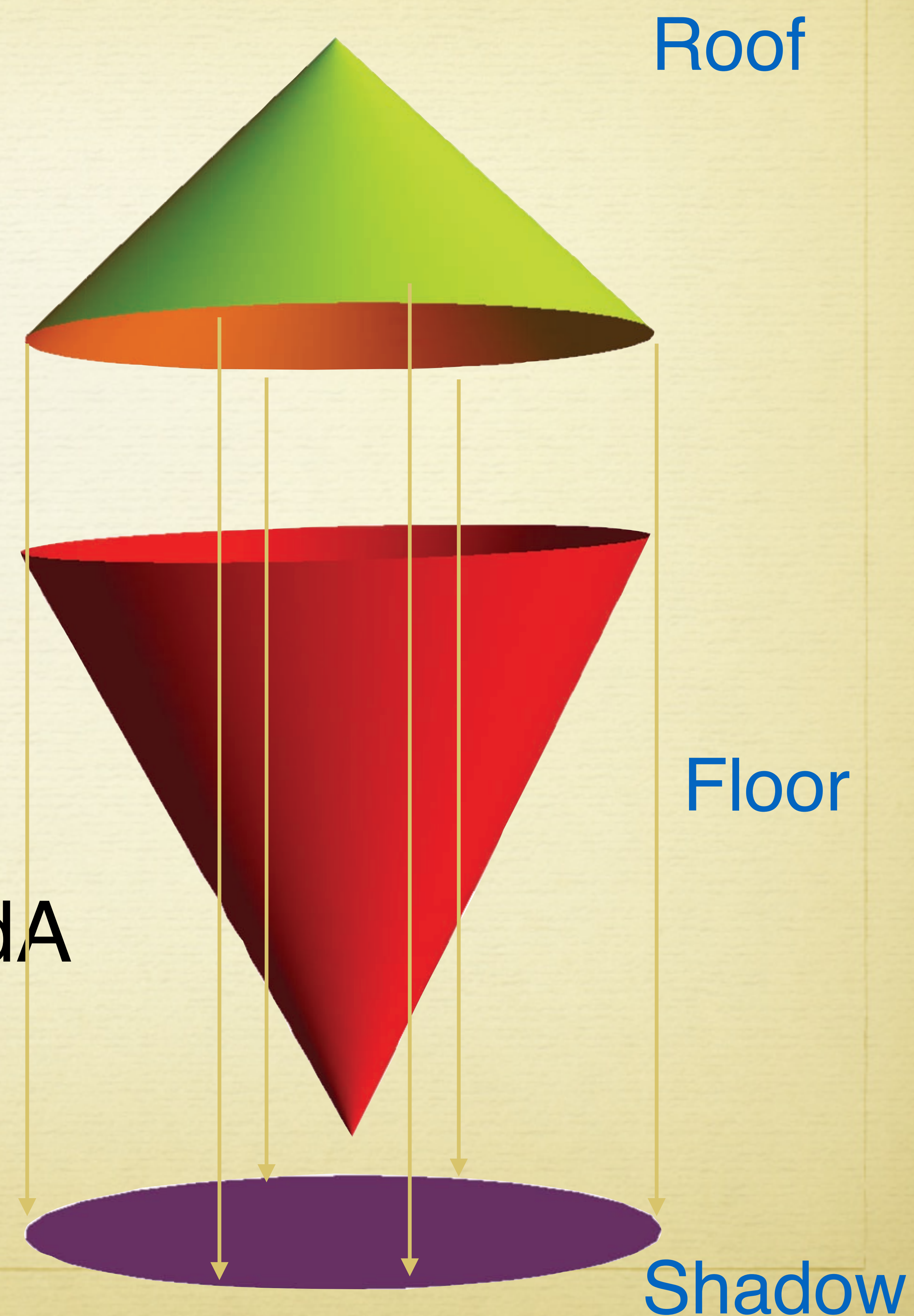
Integrate $(x^2 + y^2)^{7/2}$

over the solid

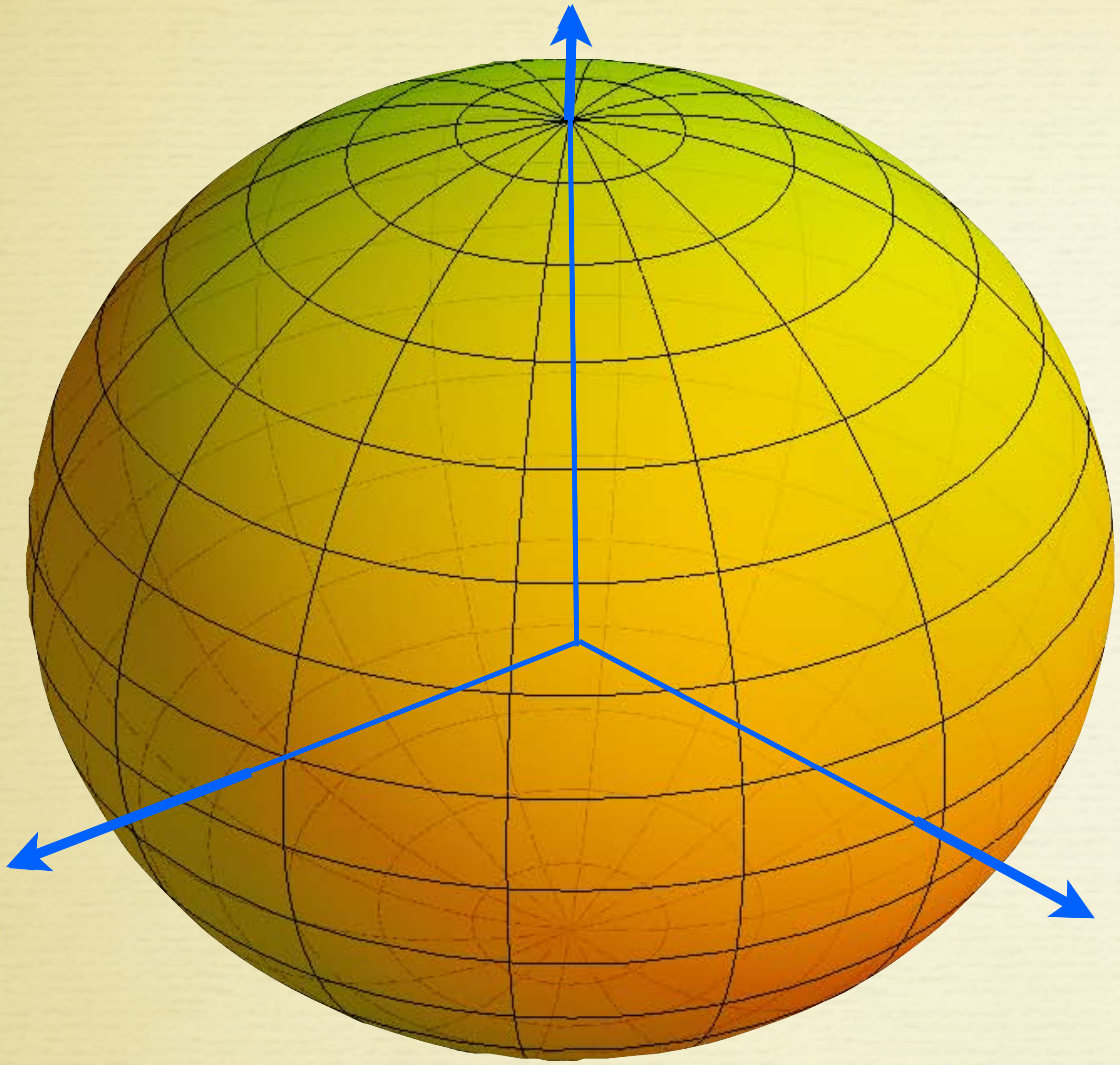
$z^2 < 1 - x^2 - y^2$

$z^2 > 2x^2 + 2y^2 - 2$

$$\iint \int_{2r^2-2}^{1-r^2} r^7 \; r \; dz \; dA$$



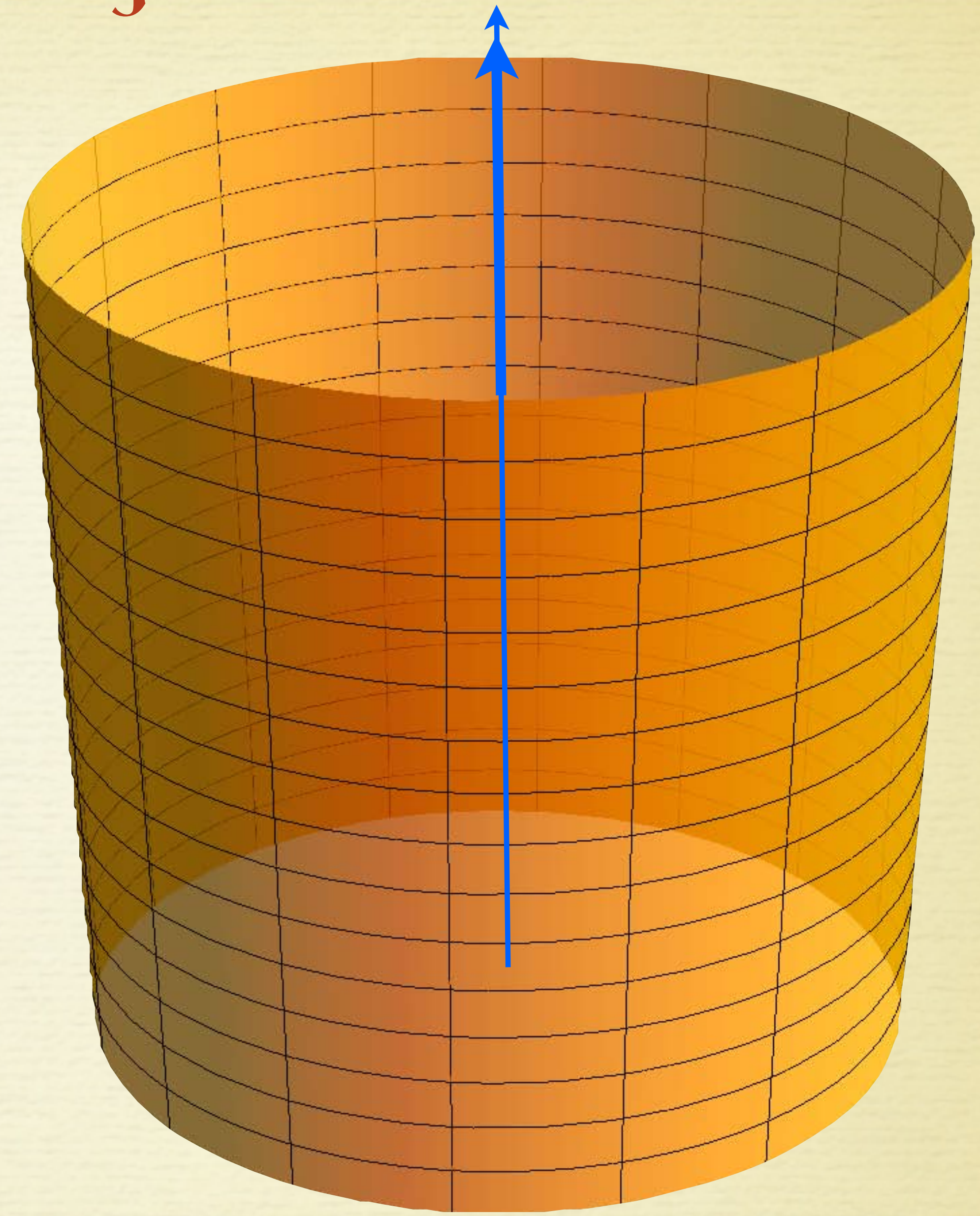
Spherical Coordinates



$$x = \varrho \sin(\phi) \cos(\theta)$$
$$y = \varrho \sin(\phi) \sin(\theta)$$
$$z = \varrho \cos(\phi)$$

$$\varrho^2 \sin(\phi)$$

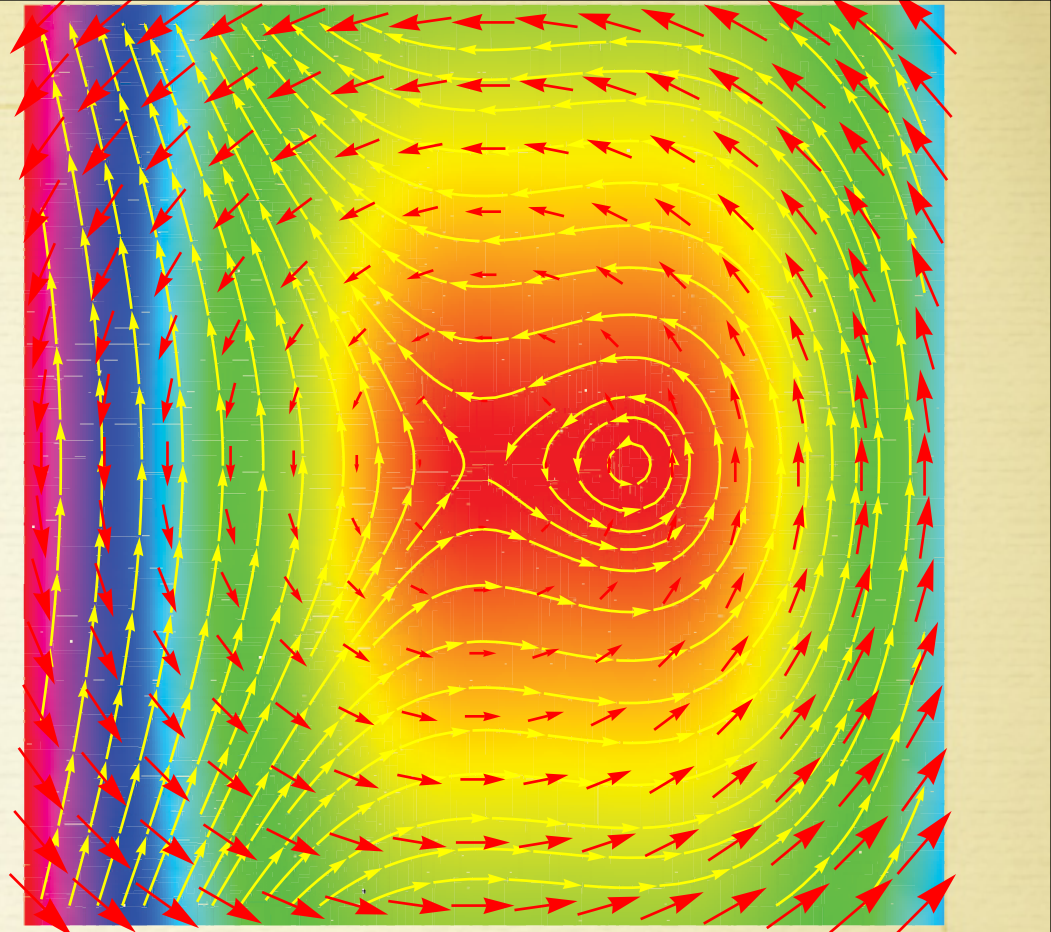
Cylindrical Coordinates



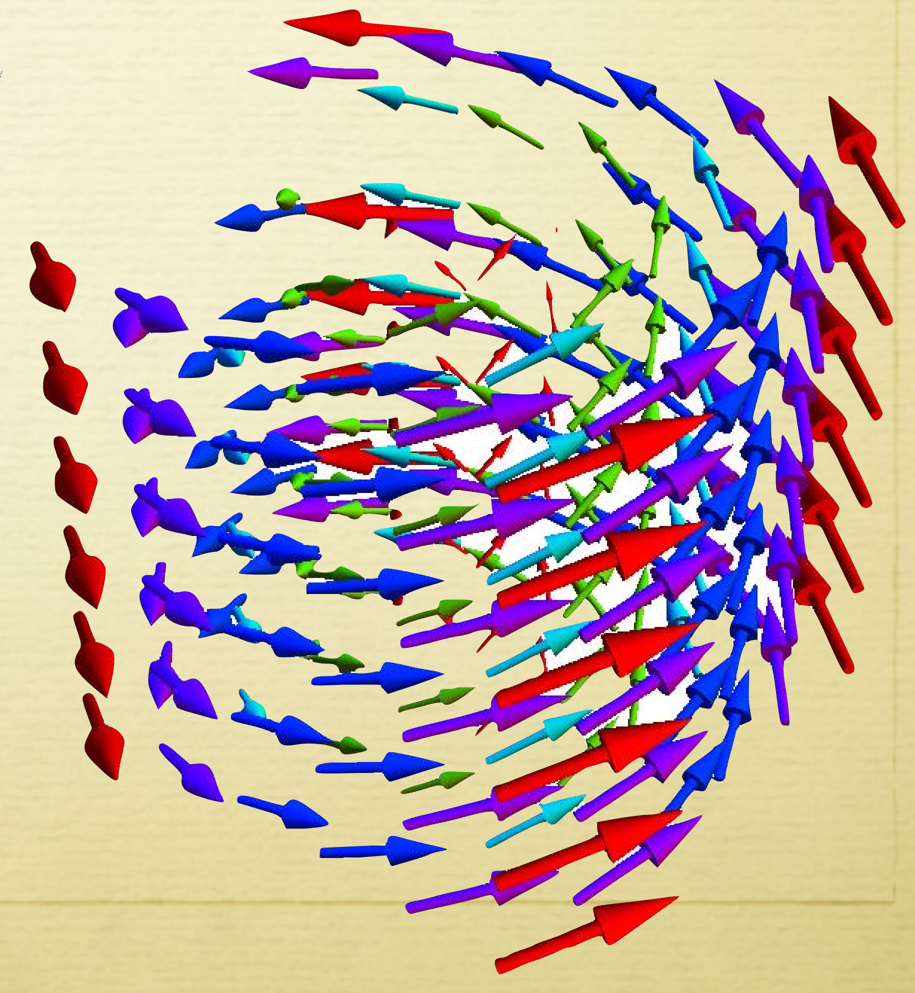
$$x = r \cos(\theta)$$
$$y = r \sin(\theta)$$
$$z = z$$

$$r$$

Vector Fields



$$F = \langle P, Q \rangle$$



$$F = \langle P, Q, R \rangle$$

$Q_x - P_y$ Curl in 2D

$P_x + Q_y$ Div in 2D

$\langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$ Curl in 3D

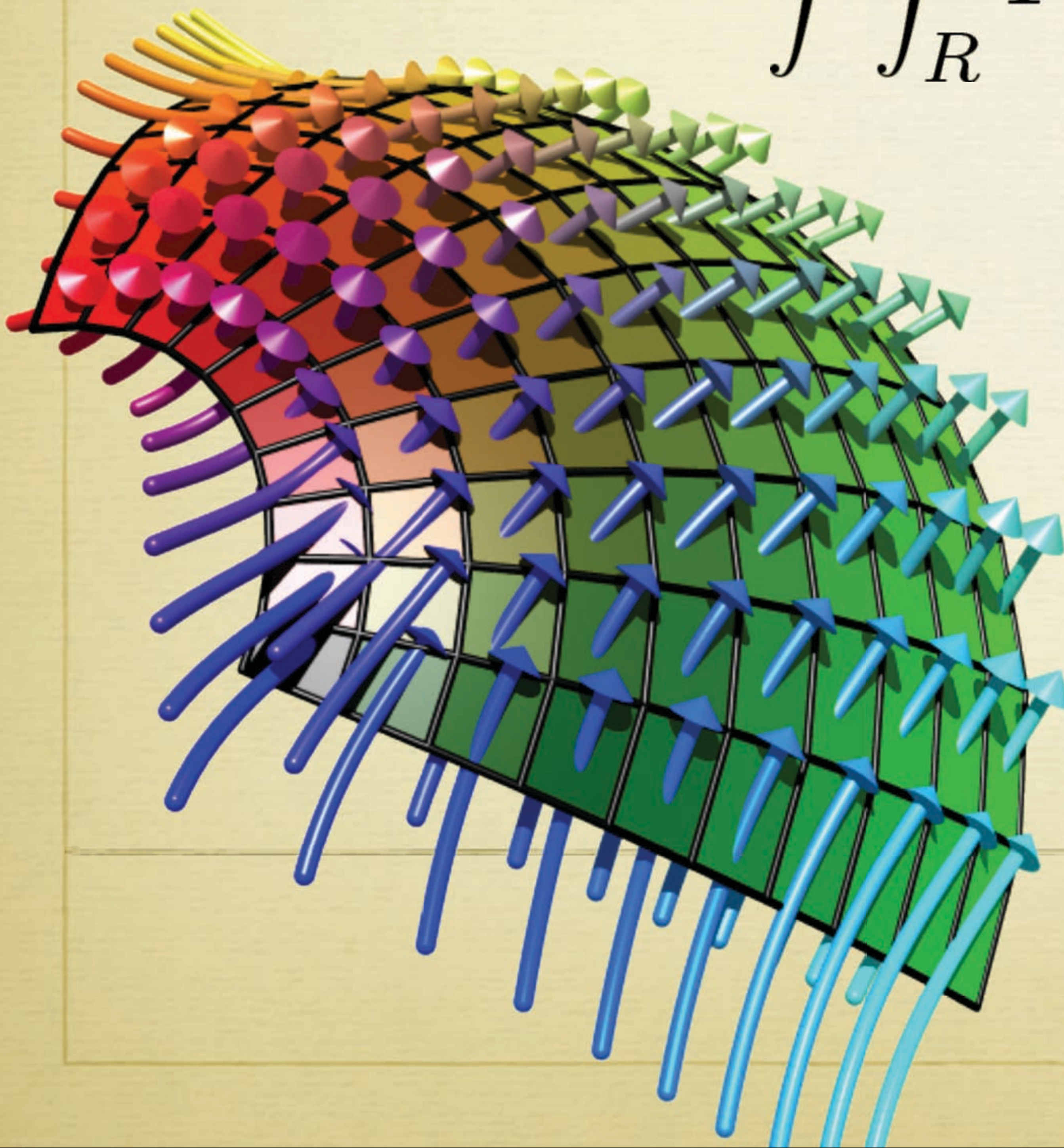
$P_x + Q_y + R_z$ Div in 3D

Div and Curl

Flux Integrals

$$\int\int_S \vec{F} \cdot d\vec{S} =$$

$$\int\int_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, du dv$$



Measures The Amount of field passing through S in Unit time.

$$\int_C \vec{F} \cdot \vec{ds} =$$

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

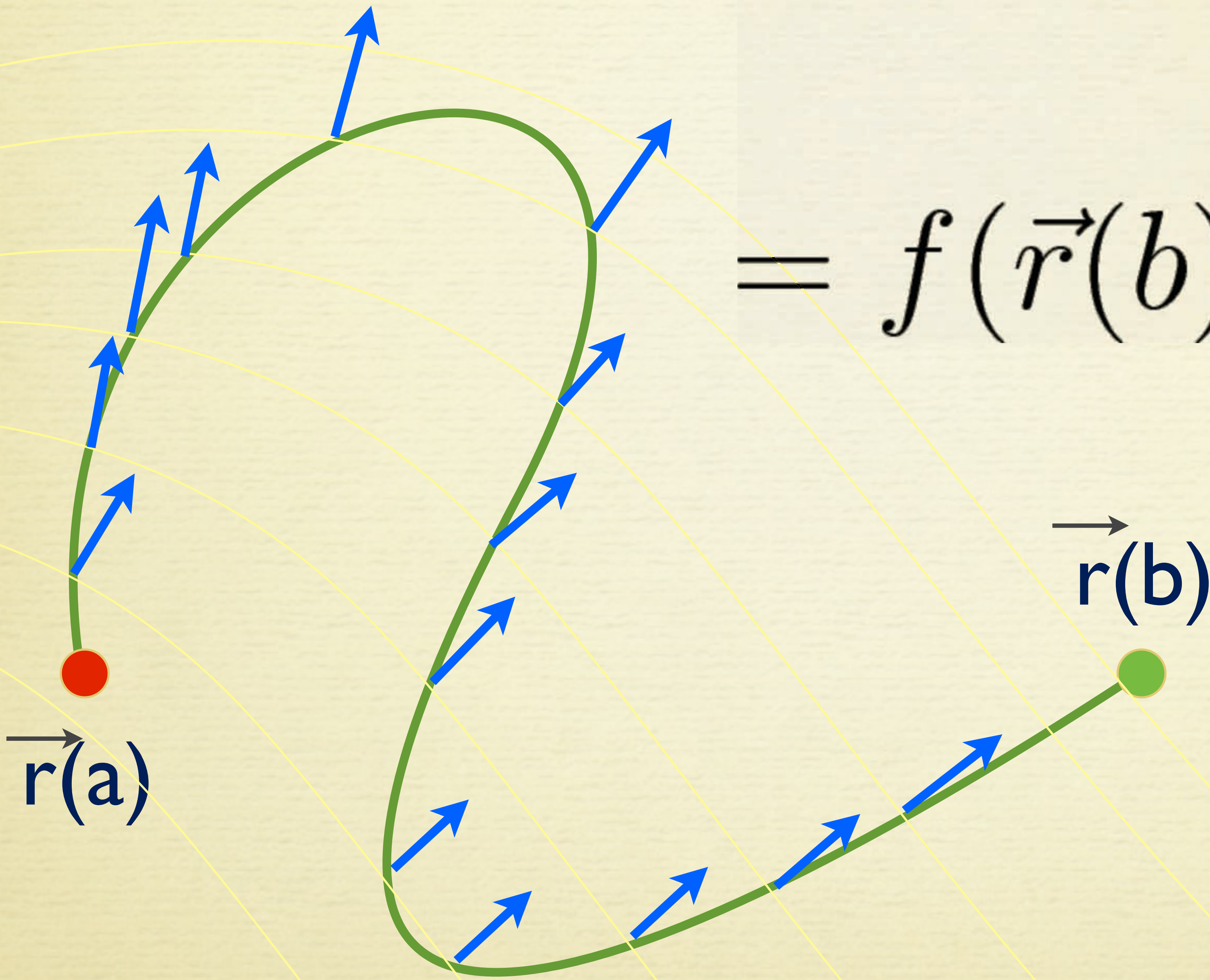
$$\int\int_S \vec{F} \cdot d\vec{S} =$$

$$\int\int_R \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, dudv$$

These two integrals is all we need.

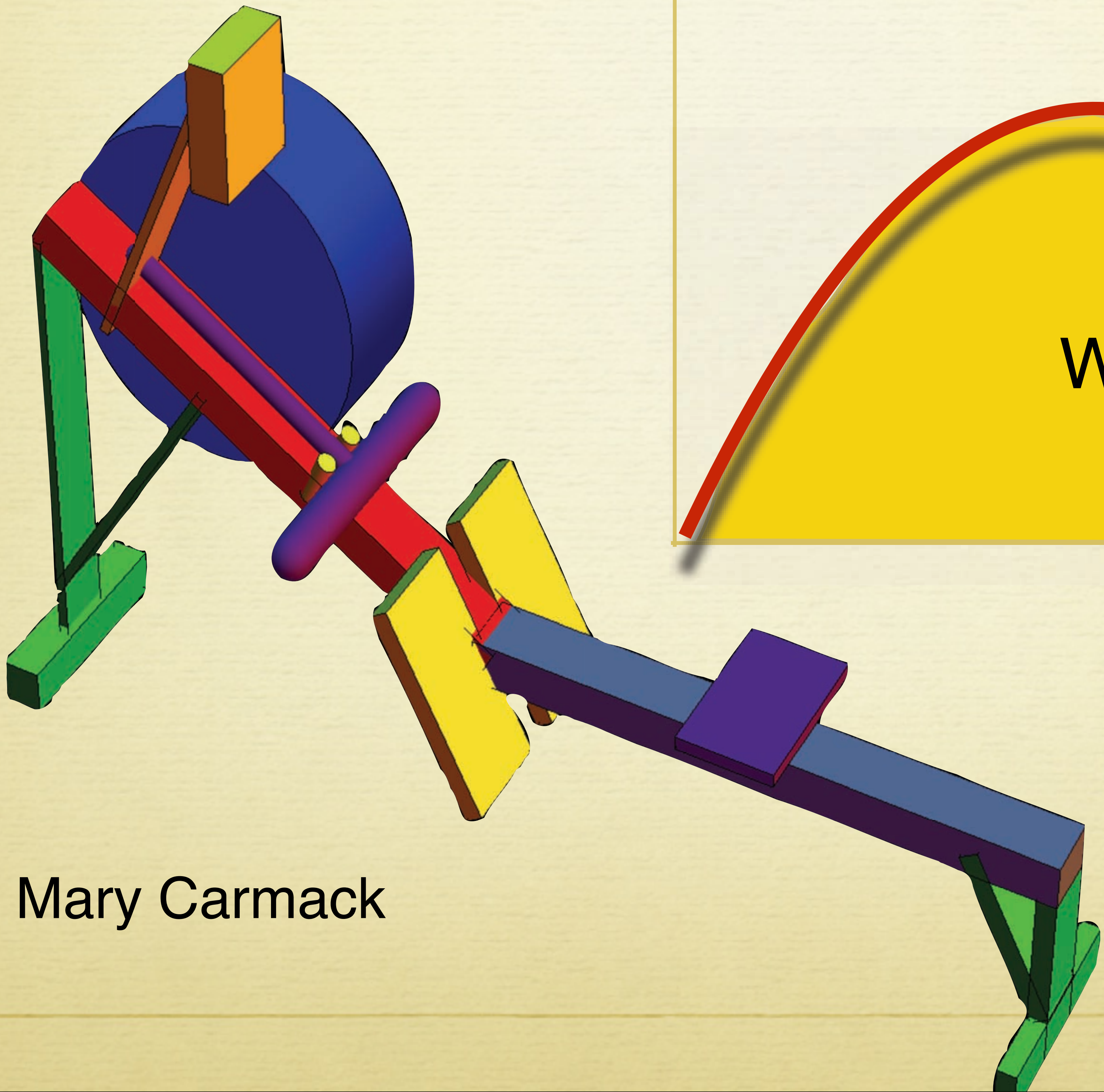
$$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

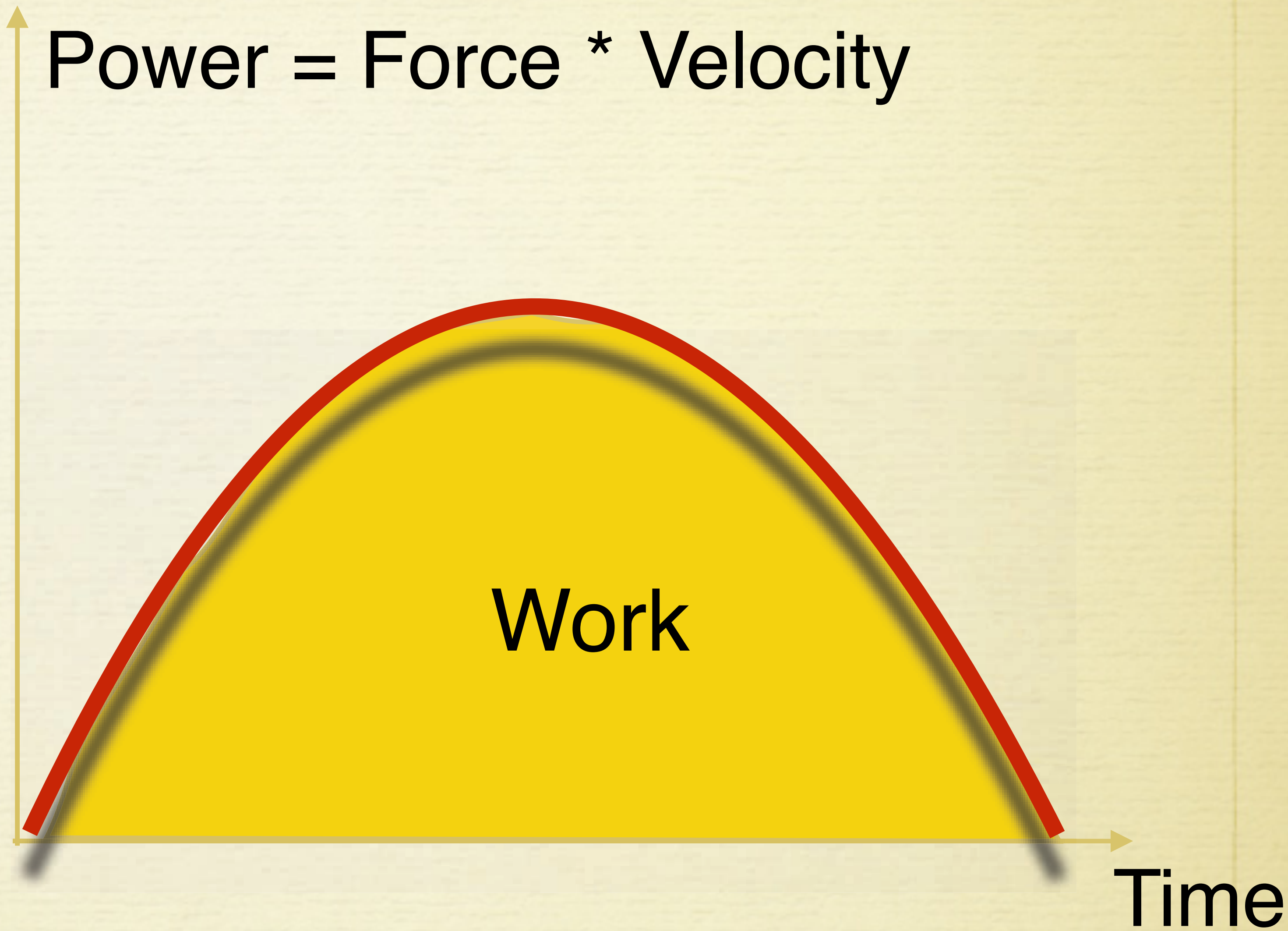
$$= f(\vec{r}(b)) - f(\vec{r}(a))$$



Fundamental Theorem of Line integrals

Power





Mary Carmack

The nano car is exposed to a force field

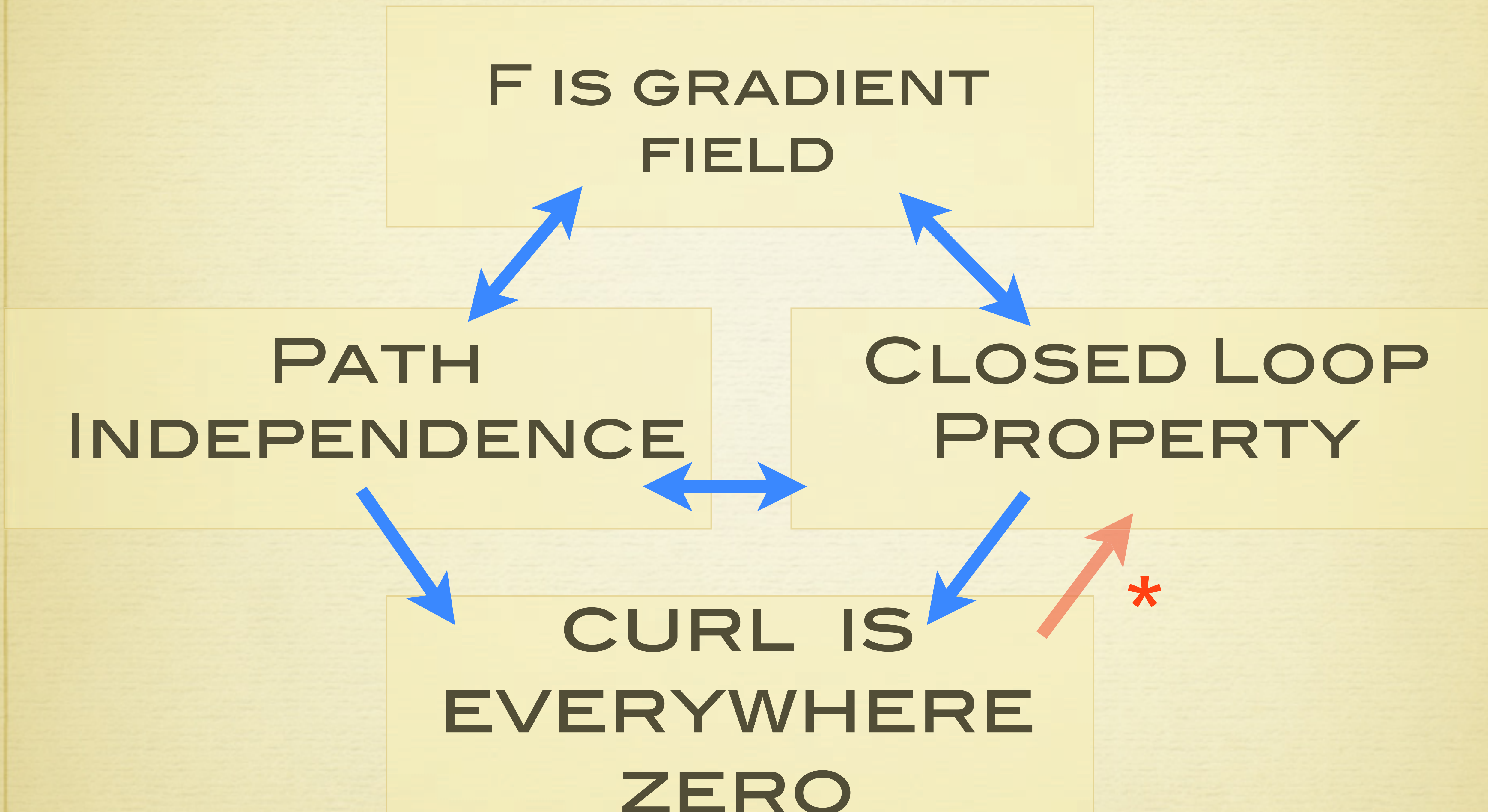
$$\vec{F}(x,y,z) = \langle yz\cos(xyz), xz\cos(xyz), xy\cos(xyz)+x \rangle$$

from the surface and pushed along a path

$$\vec{r}(t) = \langle t, \cos(t), \sin(t) \rangle$$

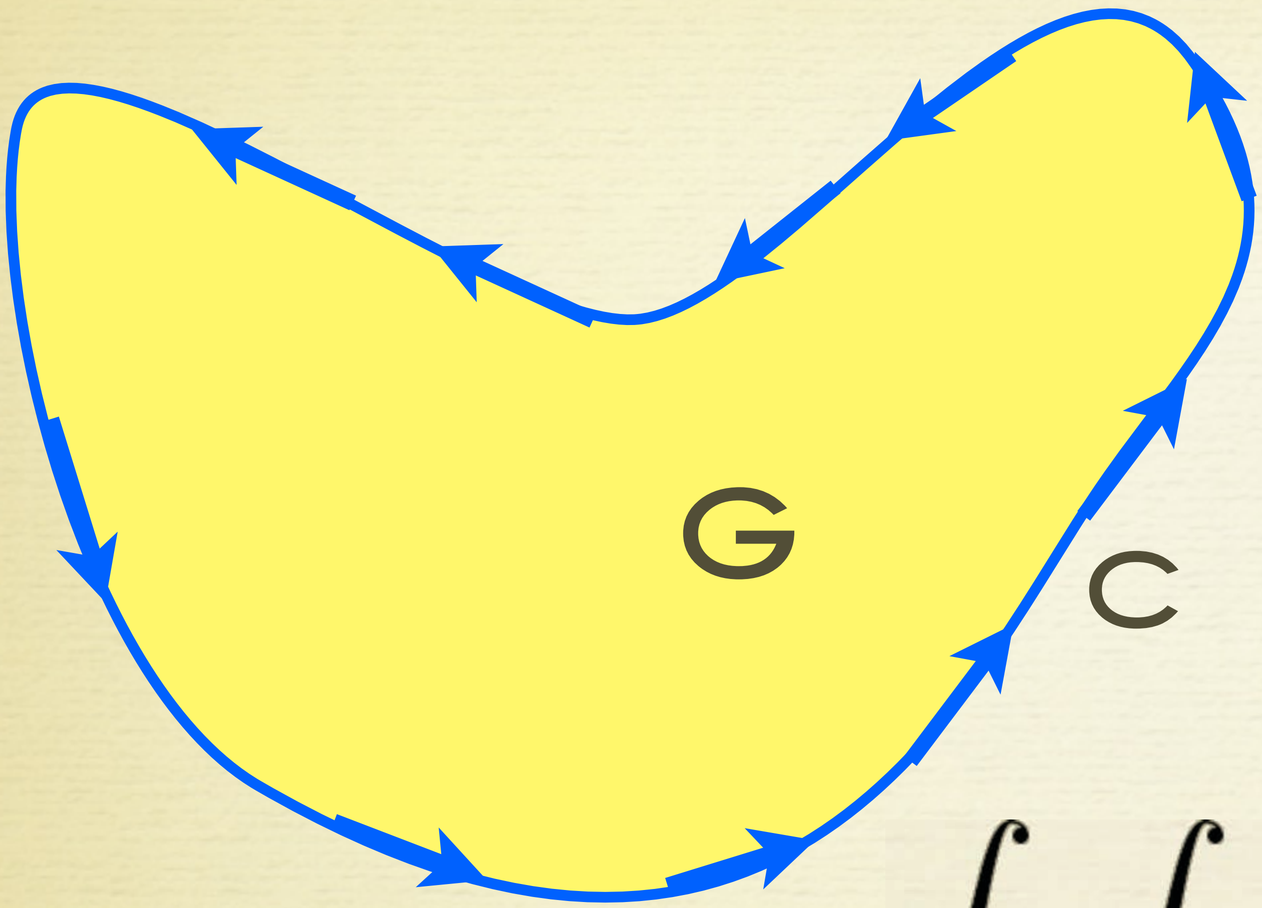
where t goes from 0 to π. What work is done on the car?

Problem



Properties of Gradient fields

* IF DEFINED IN SIMPLY CONNECTED REGION

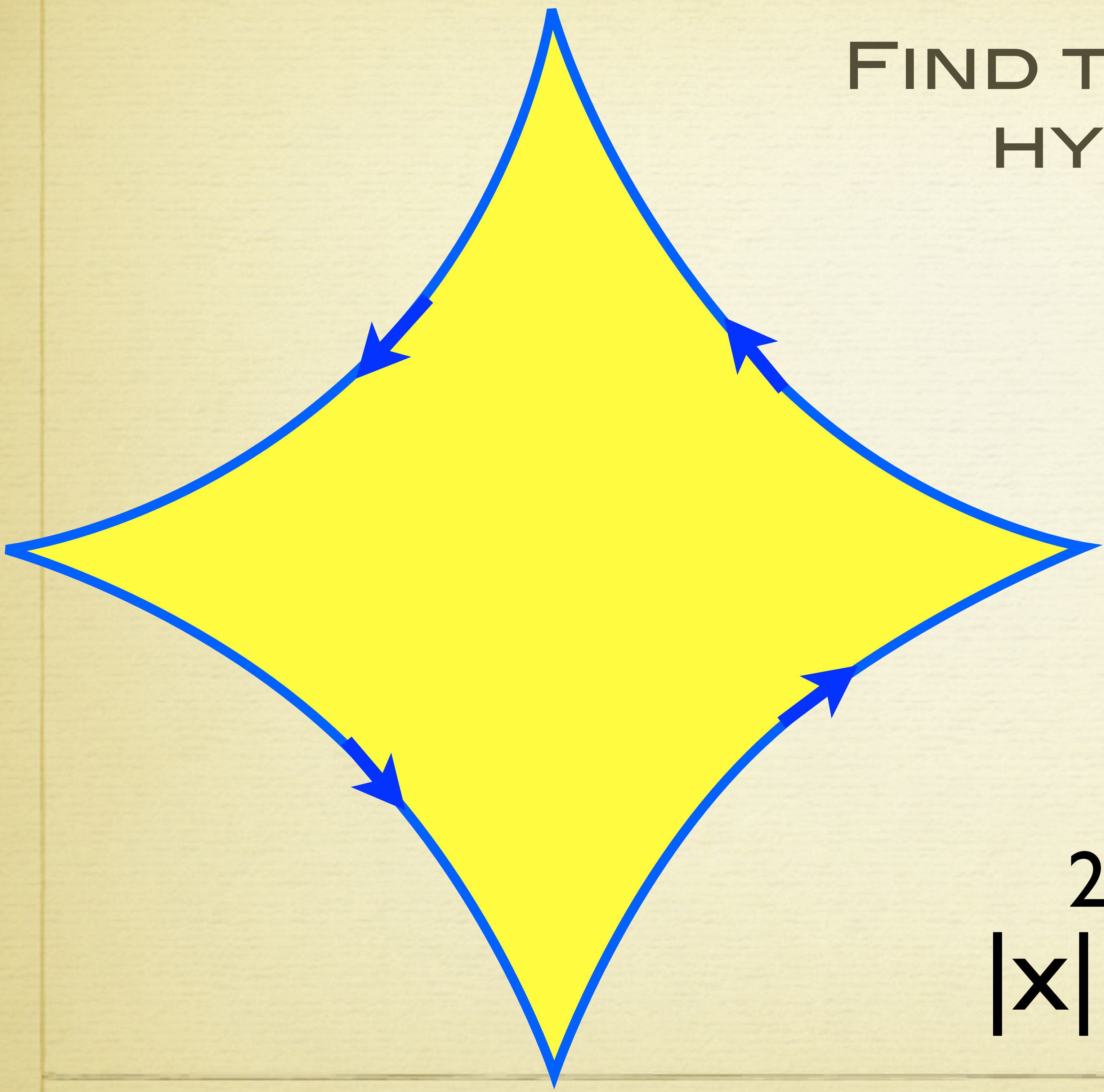


$$\int\int_G \text{curl}(\vec{F})\ dA$$

$$= \int \vec{F}\ \vec{ds}$$

Greens Theorem

Find the area of the hypercylcoid



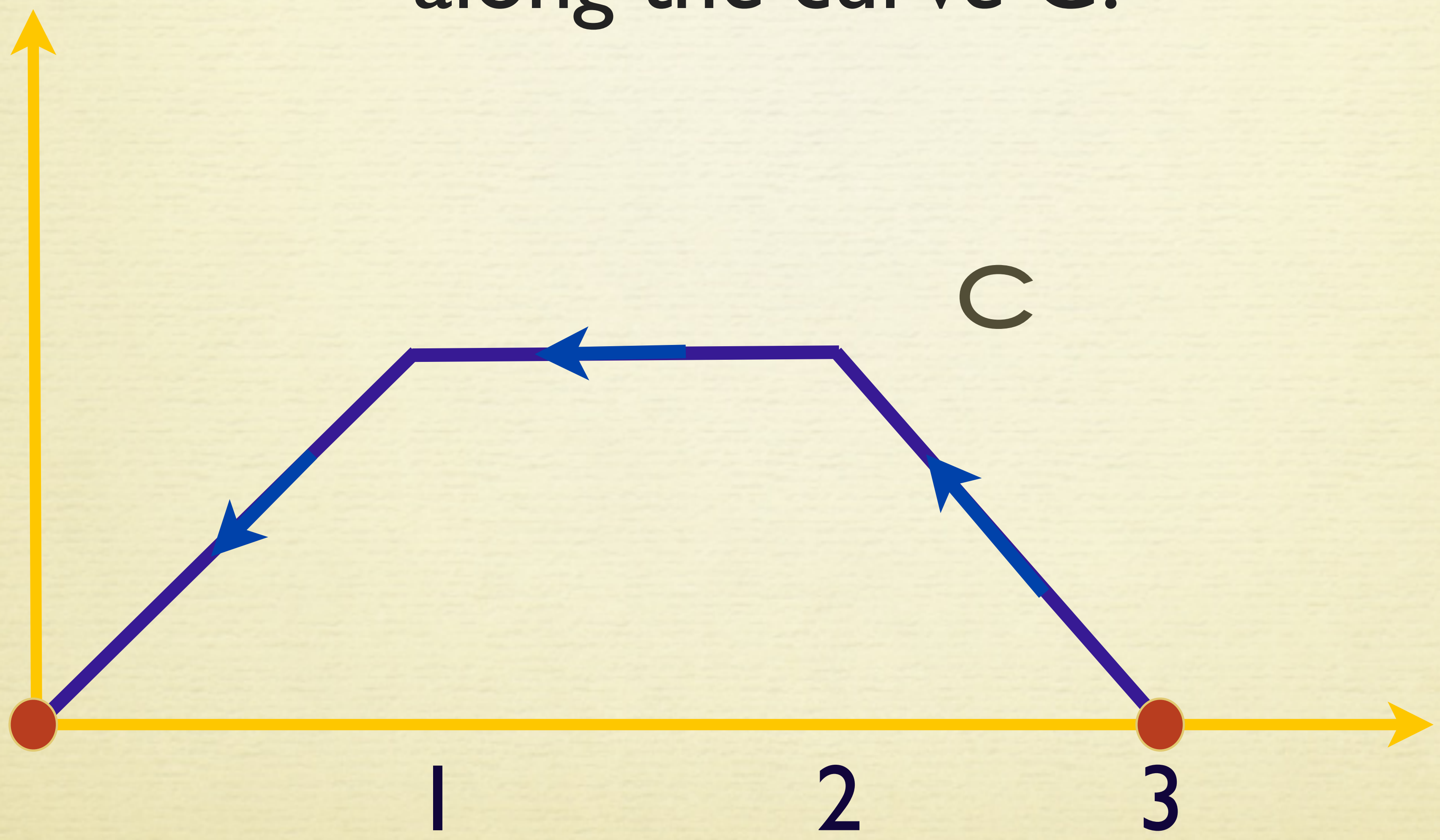
$$|x|^{2/3} + |y|^{2/3} < 1$$

Problem

Find the line integral of the vector field

$\vec{F}(x,y) = \langle y^2\cos(x) + 3, x + 2y\sin(x) \rangle$

along the curve C.



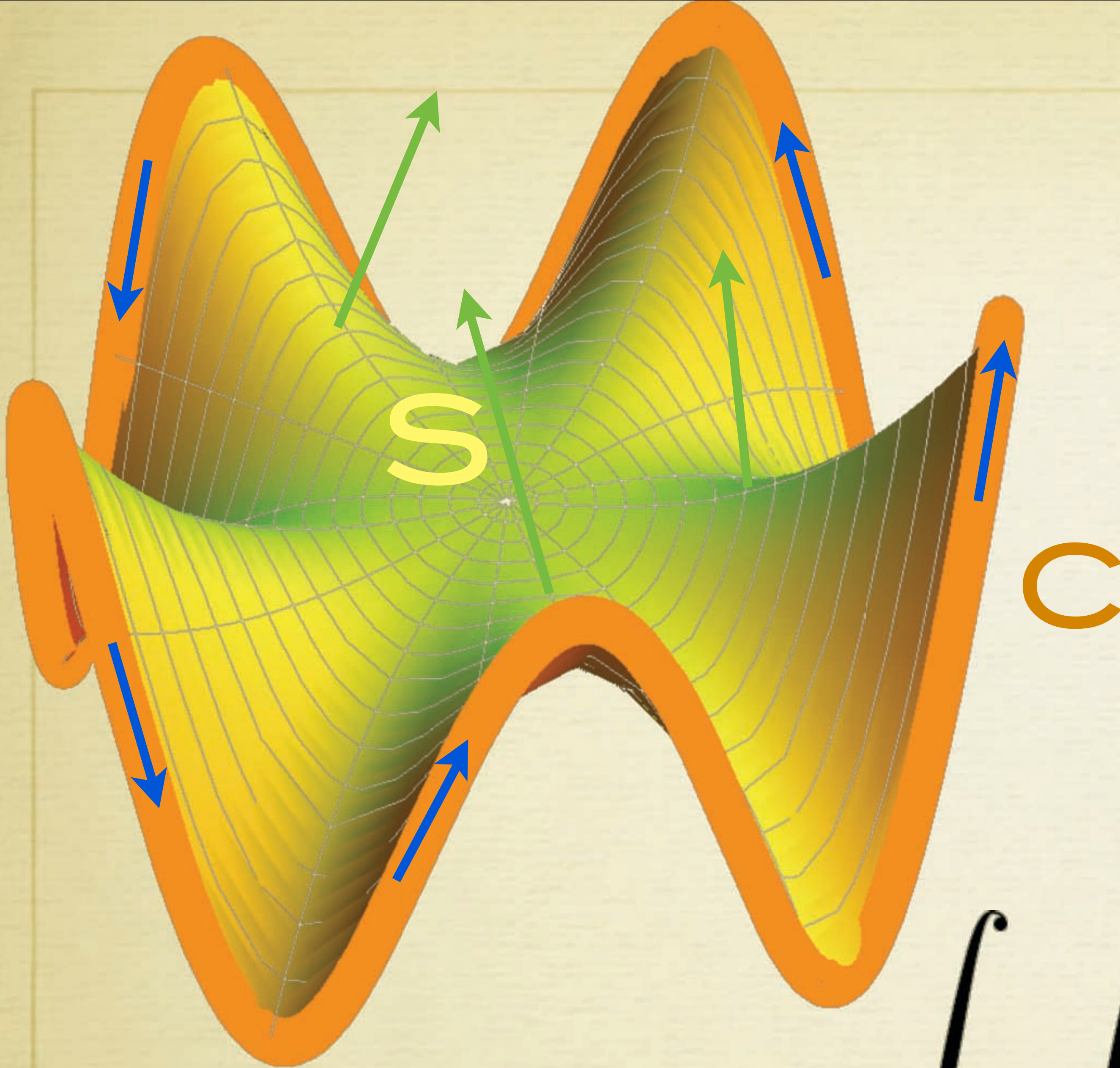
Problem

John Green paid £70 a year. After a number of years John apparently relinquished direct control of the farm to his younger brother Robert and went to farm in nearby East Bridgford. However, he maintained the lease and when he made his will[5] in 1818 he stated that "my ancestors of the name of Green have been tenants to such farm for nearly five centuries".

The farm cannot have been large enough to have provided a good livelihood for all three brothers and the Mathematician's father, as the youngest of the three, was required to make his own way in the world. Shortly after his father John's death, George Green Senior, as a youth of around 15 years, left Saxondale to take up an apprenticeship in Nottingham. His family evidently sought to do him well. They apprenticed[6] him as a baker, a trade which they must have considered to have fair prospects since they were willing to pay the master the high introductory fee of 10 guineas. Masters in the framework-knitting trade, by contrast, required only a token fee of one penny at this time. The apprenticeship, for a term of 7 years commencing on 16th July 1774, was to Robert Hill, who was a Burgess of the town. The Burgesses or Freemen, a minority of the town's population, enjoyed a number of privileges, including the right to vote in Parliamentary elections and the possibility of becoming a member of the Council, the

The farmhouse on the site of the Green's family farm at Saxondale. The present house appears to be early nineteenth century.

Oakland's Mill, Sneinton - a post mill of the type replaced by the Green's tower mill. Note the tail post. The sail arrangement was the same as on the Green's Mill. Nottinghamshire Local Studies Library 1005.



$$\int\int_S \operatorname{curl}(\vec{F}) \cdot d\vec{S}$$

$$= \int_C \vec{F} \cdot \vec{ds}$$

Stokes Theorem

Problem

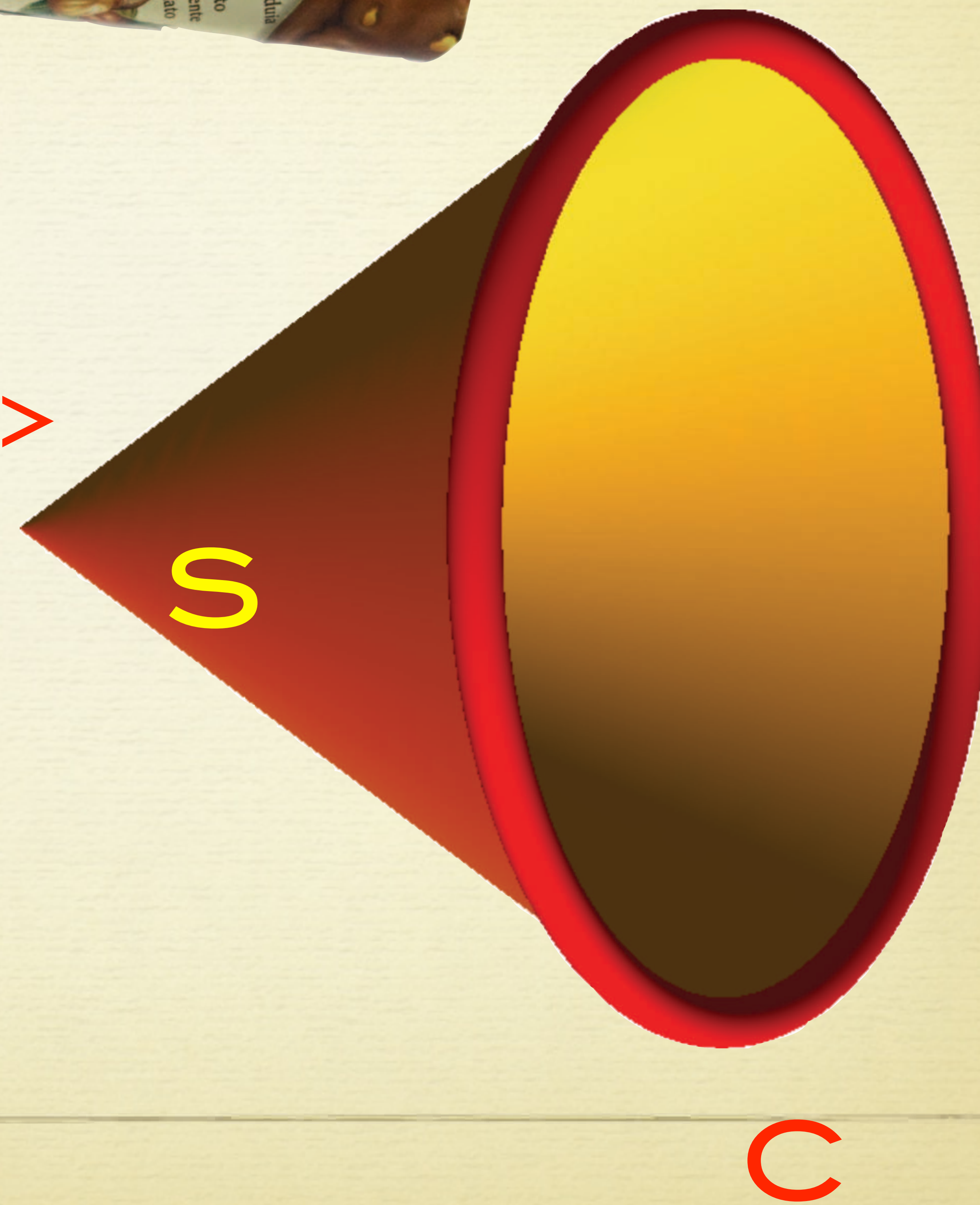
FIND THE LINE INTEGRAL OF

$\vec{F}(x,y,z) = \langle x, y, xy \;\; \rangle$

ALONG THE BOUNDARY OF

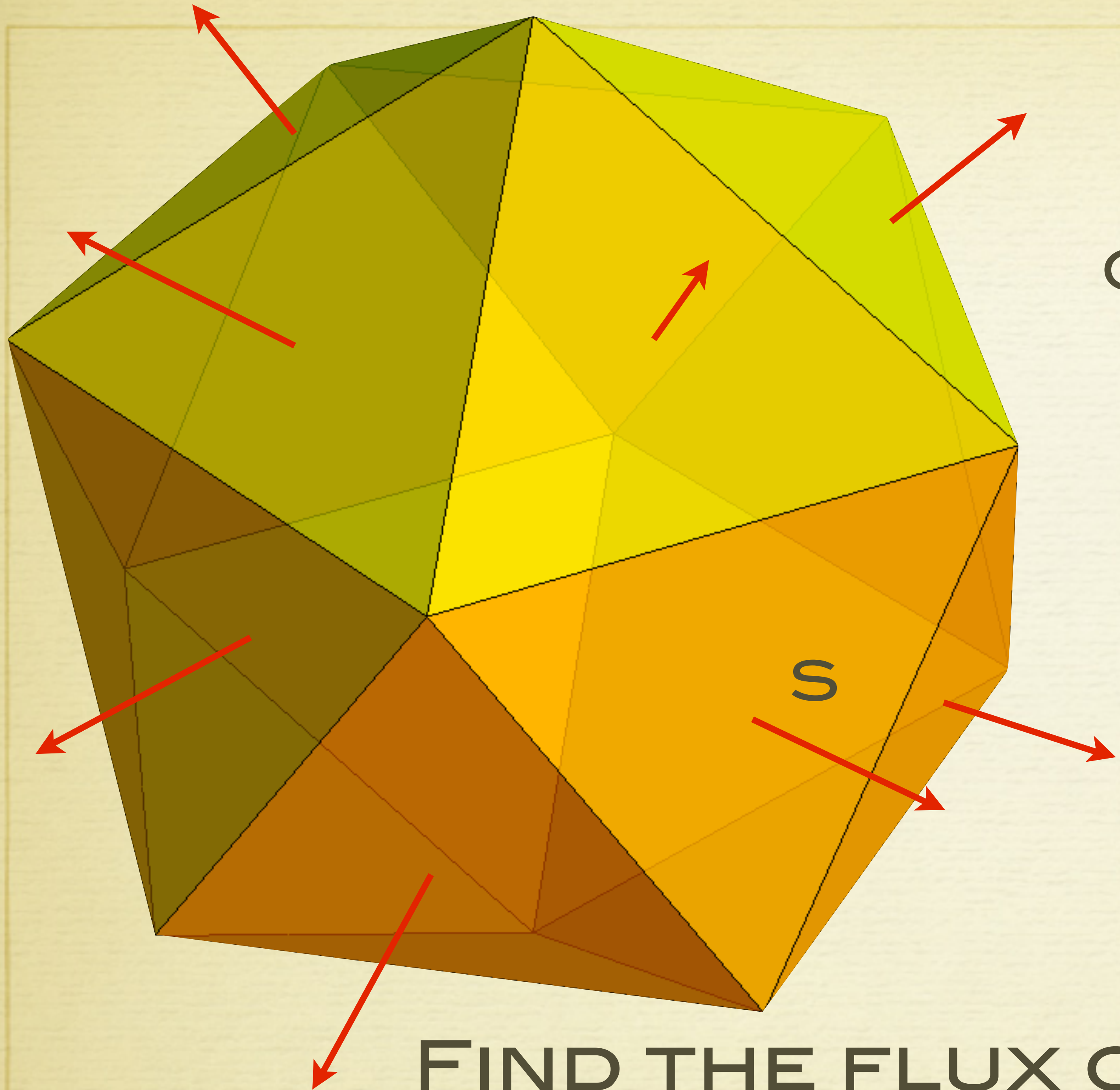
$\vec{r}(u,v) =$

$\langle v\cos(u),\ v,\ v\sin(u) \rangle$



$0 \le u < 2\pi$

$0 \le v \le 1$

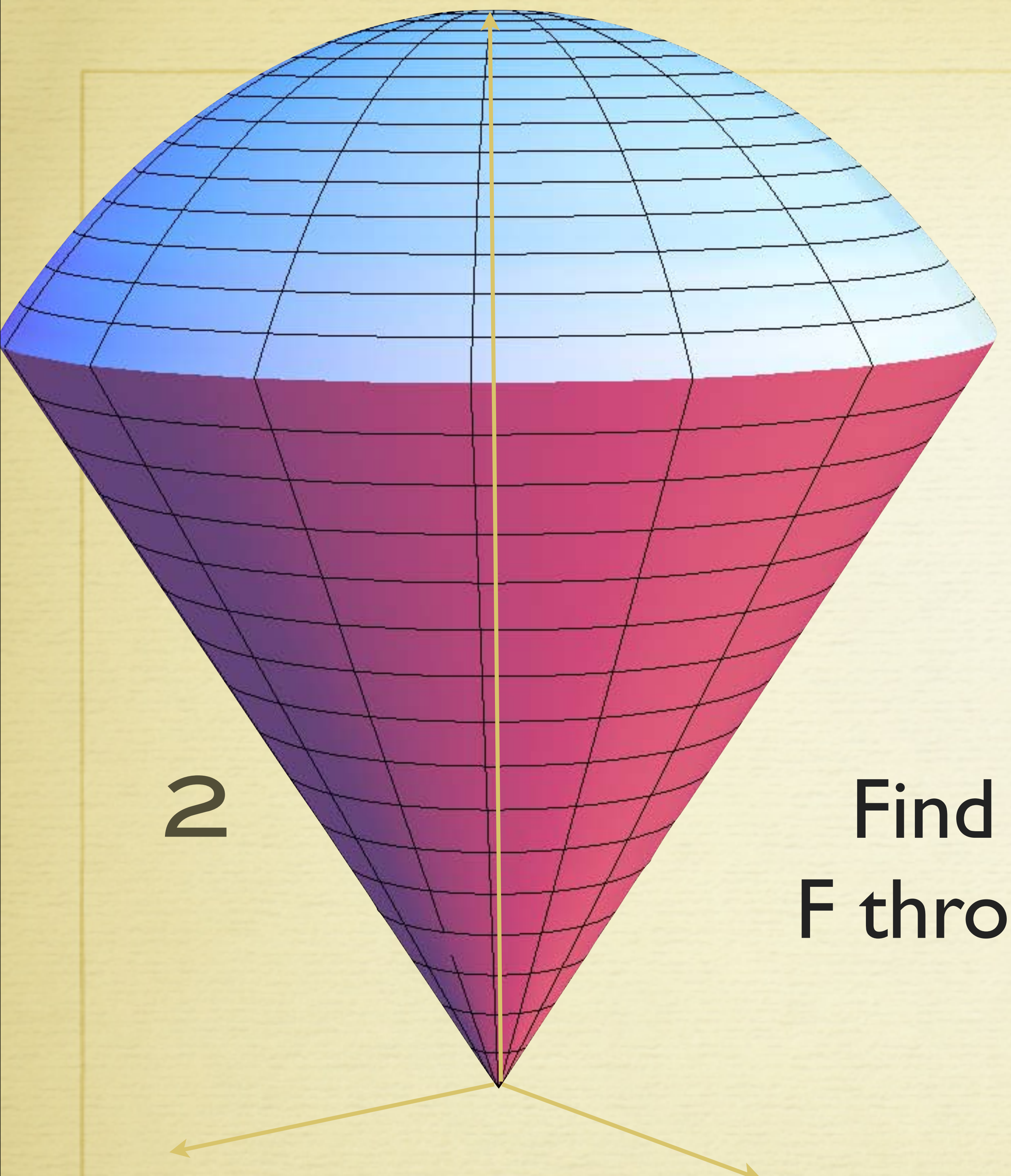


S: side length 1
Centered at origin

Find the flux of the vector field

$$\vec{F}(x,y,z) = \langle \sin(y\,z), \cos(x\,z), \tan(x\,y) \rangle$$

Problem 9 Through the Icosahedron



$$x^2 + y^2 + z^2 \leq 4$$
$$z \geq 0$$
$$x^2 + y^2 \leq z^2$$

Find the flux of the vector field F through the the boundary of the region.

$$\vec{F}(x,y,z) = \langle x^3 - \sin(y), y^3 + 1, \sin(3 + y\sin(x)) \rangle$$

1

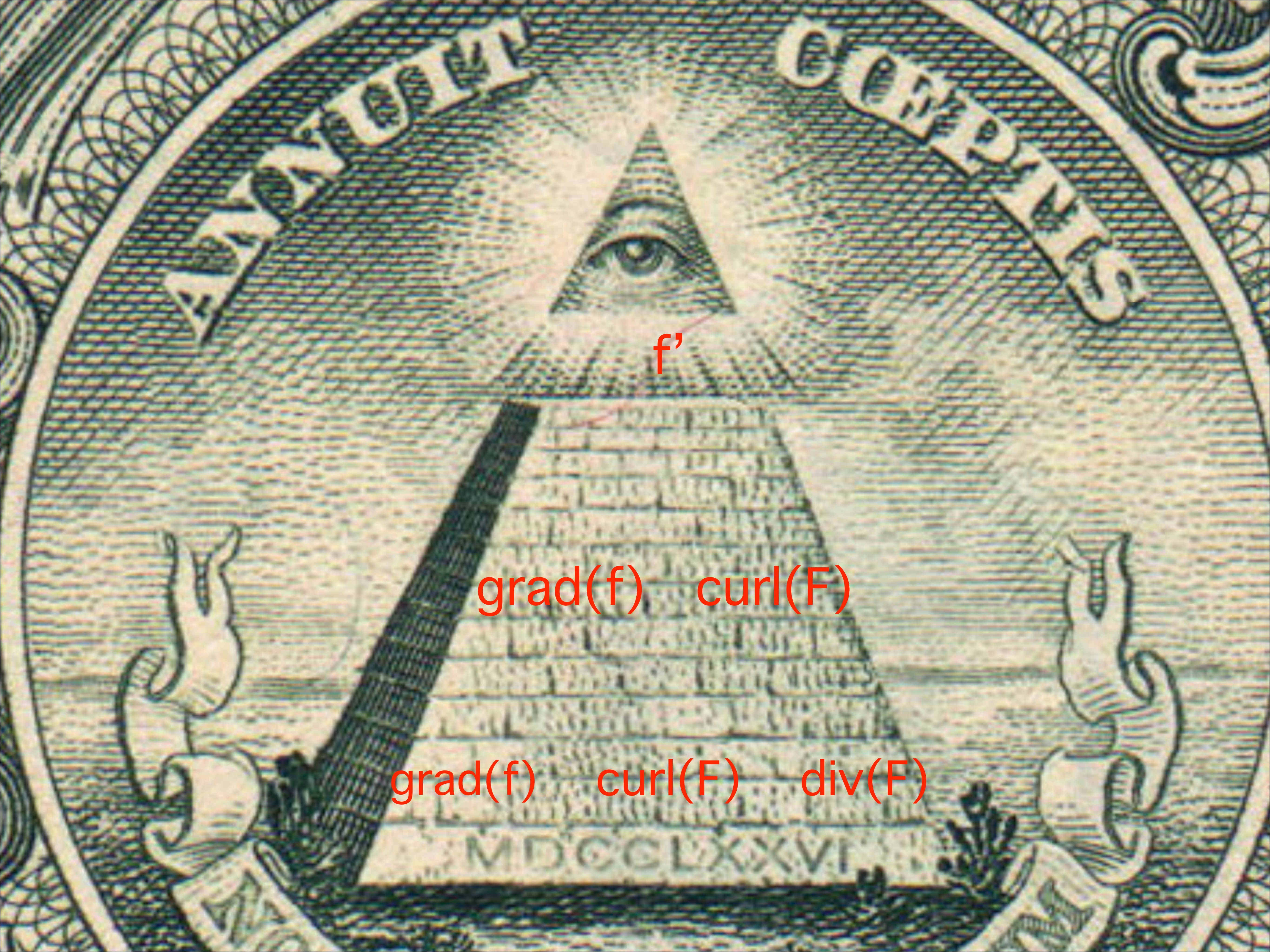
1 f’ 1

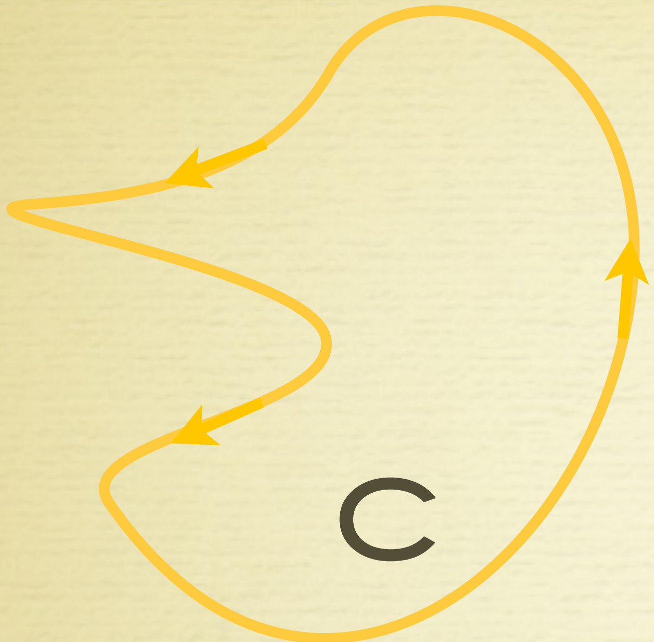
1 grad(f) 2 curl(F) 1

1 grad(f) 3 curl(F) 3 div(F) 1

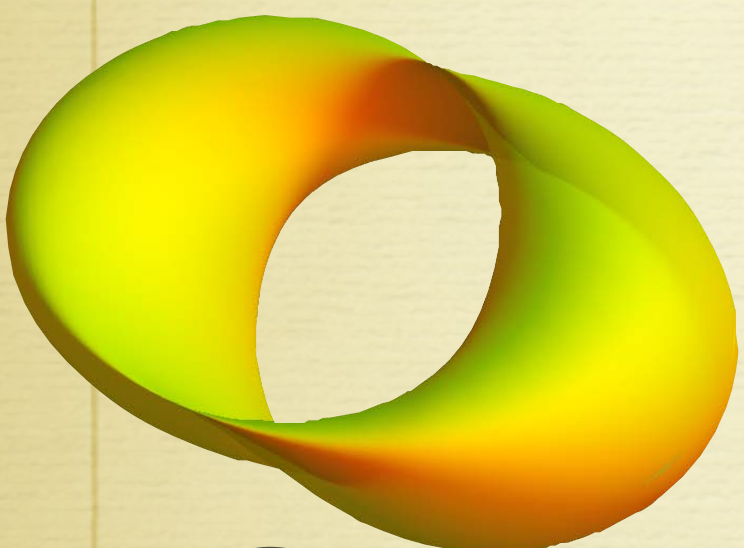
1 4 6 4 1

Derivatives


ANNUIT CŒPTIS
f'
grad(f) curl(F)
grad(f) curl(F) div(F)
MDCCLXXVI



$$\int_C \mathrm{grad}(f)\,(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = 0$$



S

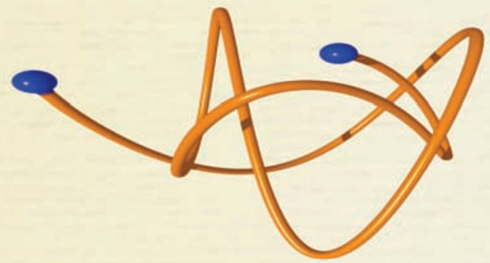
$$\iint_S \mathrm{curl}(\vec{F})(\vec{r}(u,v))\cdot\vec{r}_u \times \vec{r}_v\,du\,dv = 0$$

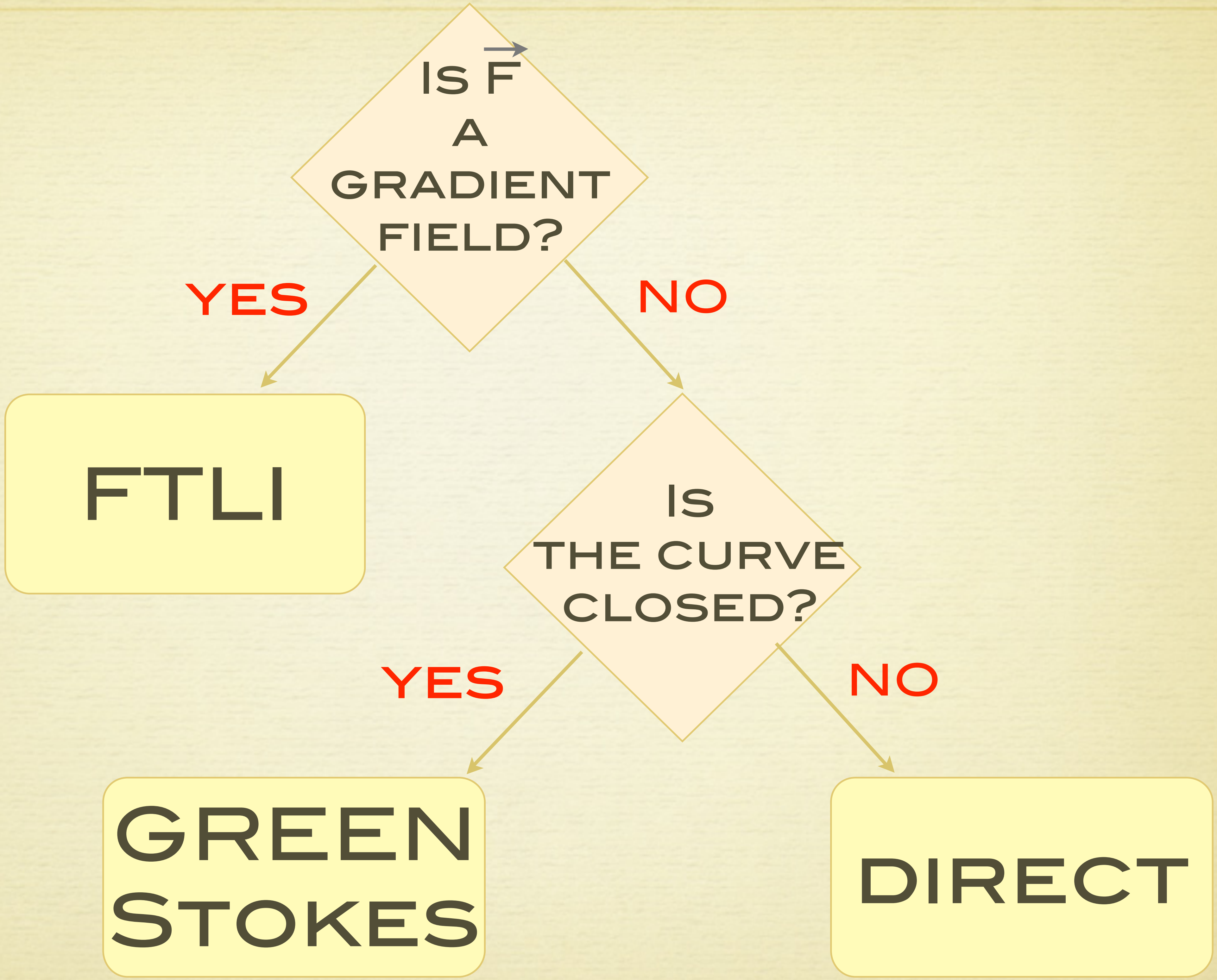
identities

$$\mathrm{curl}\ \mathrm{grad}\ (f) = 0$$

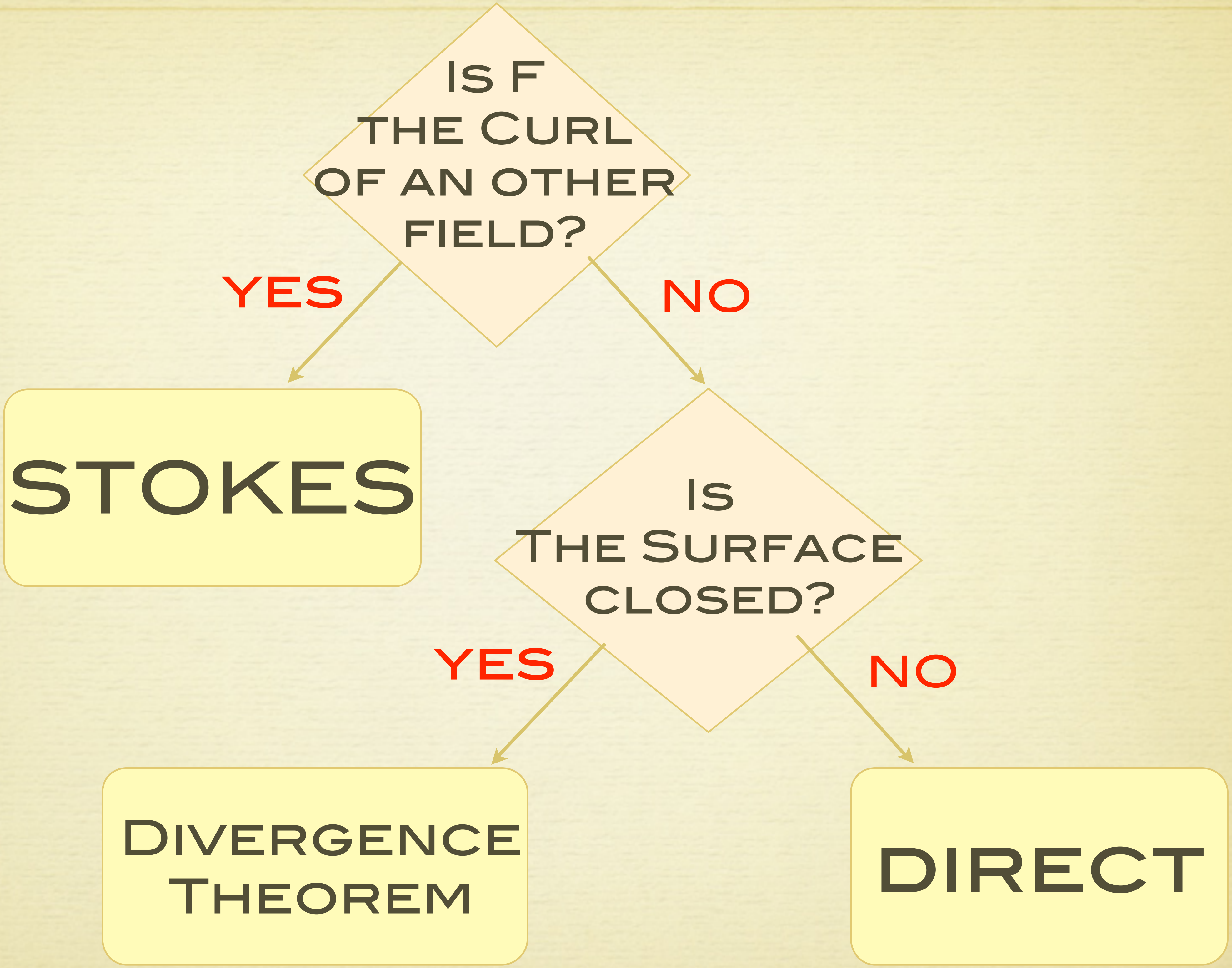
$$\mathrm{div}\ \mathrm{curl}\ (\vec{F}) = 0$$

Overview Theorems

			
dim=1	FTC		
dim=2	FTLI	Green	
dim=3	FTLI	Stokes	Gauss



Decision Tree for Line integrals



Decision Tree for Flux integrals

CARL FRIEDRICH GAUSS

1777-1855

GEORGE GREEN

1793-1841

GEORGE STOKES

1819-1903

JEAN MARY AMPERE

1775-1846

MIKHAEL OSTROGRADSKY

1801-1862

AUGUSTINE CAUCHY

1789-1857

FTLI 1840

GAUSS THEOREM

GREEN THEOREM

STOKES THEOREM