

Name:

MWF 9 Oliver Knill
MWF 9 Chao Li
MWF 10 Gijs Heuts
MWF 10 Adrian Zahariuc
MWF 10 Yihang Zhu
MWF 11 Peter Garfield
MWF 11 Matthew Woolf
MWF 12 Charmaine Sia
MWF 12 Steve Wang
MWF 14 Mike Hopkins
TTH 10 Oliver Knill
TTH 10 Francesco Cavazzani
TTH 11:30 Kate Penner
TTH 11:30 Francesco Cavazzani

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
14		10
Total:		150

- 1)

T	F
---	---

 There are two unit vectors $\vec{v}$, $\vec{w}$ for which the sum $\vec{v} + \vec{w}$ has length $1/3$.
- 2)

T	F
---	---

 For any three vectors, we have $|(\vec{u} \times \vec{v}) \times \vec{w}| = |(\vec{v} \times \vec{w}) \times \vec{u}|$.
- 3)

T	F
---	---

 Denote by $d(P, L)$ the distance from a point P to a line L in space. For any point P and any two lines L, K in space, we have $d(P, L) + d(P, K) \geq d(L, K)$.
- 4)

T	F
---	---

 For any three vectors $\vec{u}, \vec{v}, \vec{w}$, the relation $|\vec{u} \times (\vec{v} \times \vec{w})| \leq |\vec{u}||\vec{v}||\vec{w}|$ holds.
- 5)

T	F
---	---

 If $\vec{r}(t)$ has speed 1 and curvature 1 everywhere, then $\vec{r}(2t)$ has constant speed 2 and constant curvature $1/2$ everywhere.
- 6)

T	F
---	---

 If the curvature of a space curve is constant 1 and the speed $|\vec{r}'(t)| = 1$ everywhere, then the acceleration satisfies $|\vec{r}''(t)| = 1$ everywhere.
- 7)

T	F
---	---

 If a vector field $\vec{F} = \langle P, Q \rangle$ has $\text{curl}(\vec{F}) = Q_x - P_y = 0$ everywhere and divergence $\text{div}(\vec{F}) = P_x + Q_y = 0$ everywhere, then $\vec{F}$ must be constant.
- 8)

T	F
---	---

 If the level curve $f(x, y) = 1$ contains both the lines $x = y$ and $x = -y$, then $(0, 0)$ must be a critical point for which $D < 0$.
- 9)

T	F
---	---

 The surface $\vec{r}(u, v) = \langle u^3 \cos(v), u^3 \sin(v), u^3 \rangle$ with $v \in [0, 2\pi)$ and $-\infty \leq u \leq \infty$ is a double cone.
- 10)

T	F
---	---

 There is a non-constant function $f(x, y, z)$ of three variables such that $\text{div}(\text{grad}(f)) = f$.
- 11)

T	F
---	---

 If $\text{curl}(\vec{F}) = \vec{F}$, then the vector field $\vec{F}$ satisfies $\text{div}(\vec{F}) = 0$ everywhere.
- 12)

T	F
---	---

 The equation $\phi = \pi/4$ in spherical coordinates defines a half plane.
- 13)

T	F
---	---

 The tangent plane of $x^3 + y^2 + z^4 = 9$ at $(0, 3, 0)$ is $y = 3$.
- 14)

T	F
---	---

 Assume (x_0, y_0) is not a critical point of $f(x, y)$. It is possible that f increases at (x_0, y_0) most rapidly in the direction $\langle 1, 0 \rangle$ and decreases most rapidly in the direction $\langle 4/5, -3/5 \rangle$.
- 15)

T	F
---	---

 Assume $\vec{F}(x, y, z)$ is defined everywhere except on the z -axis and satisfies $\text{curl}(\vec{F}) = \vec{0}$ everywhere except on the z -axis, then $\int_C \vec{F} \cdot d\vec{r} = 0$ for all curves C .
- 16)

T	F
---	---

 A point (x_0, y_0) is an extremum of $f(x, y)$ under the constraint $g(x, y) = 0$. If $D = f_{xx}f_{yy} - f_{xy}^2 > 0$, then (x_0, y_0) can not be a local maximum on the constraint curve.
- 17)

T	F
---	---

 The vector field $\vec{F}(x, y, z) = \langle x^2, y^2, z^2 \rangle$ can be the curl of another vector field $\vec{G}$.
- 18)

T	F
---	---

 If $f(x, y)$ and $g(x, y)$ are two functions and $(2, 3, 3)$ is a critical point of the function $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$, then $(2, 3)$ is a solution of the Lagrange equations for extremizing $f(x, y)$ under the constraint $g(x, y) = 0$.
- 19)

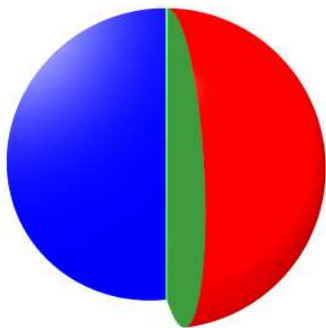
T	F
---	---

 Assume $(0, 0)$ is a global maximum of $f(x, y)$ on the disc $D = \{x^2 + y^2 \leq 1\}$, then $\int \int_D f(x, y) \, dx dy \leq \pi f(0, 0)$.
- 20)

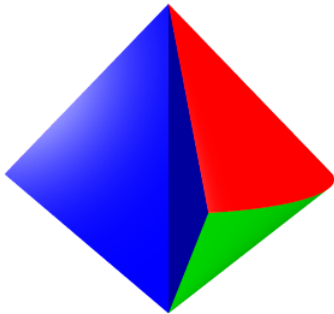
T	F
---	---

 Let C be a curve parametrized by $\vec{r}(t)$, $0 \leq t \leq 1$ for which the acceleration is constant 1. Then $\int_C \nabla f \cdot d\vec{r}$ is equal to $\int_0^1 D_{\vec{r}''(t)}(f(\vec{r}(t))) \, dt$.

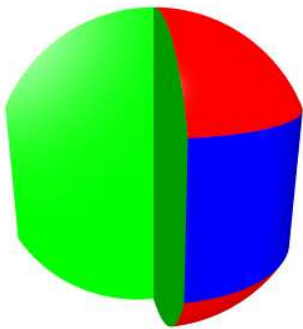
a) (4 points) Match the following triple integrals with the regions.



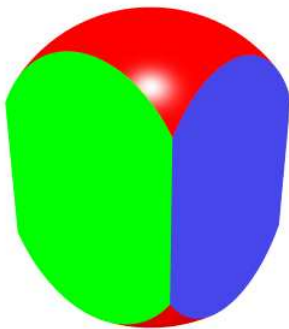
I



II



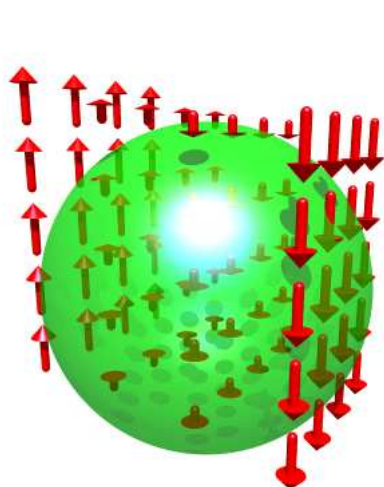
III



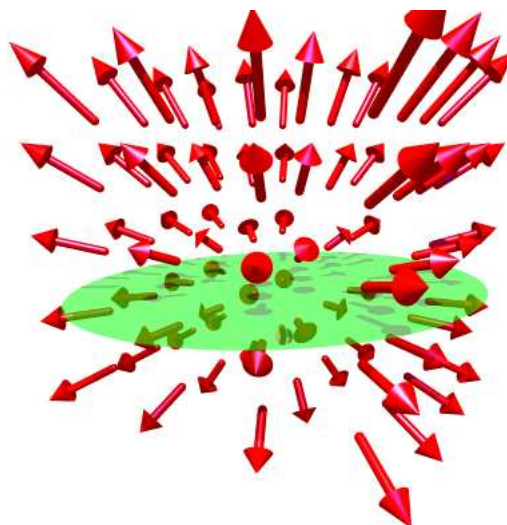
IV

Enter I,II,III,IV here	Equation
	$\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} f(r \cos(\theta), r \sin(\theta), z) r \, dz dr d\theta$
	$\int_0^{3\pi/2} \int_0^1 \int_{r-1}^{1-r} f(r \cos(\theta), r \sin(\theta), z) r \, dz dr d\theta$
	$\int_0^{3\pi/2} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} f(r \cos(\theta), r \sin(\theta), z) r \, dz dr d\theta$
	$\int_{-1}^1 \int_{-1}^1 \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} f(x, y, z) \, dz dy dx$

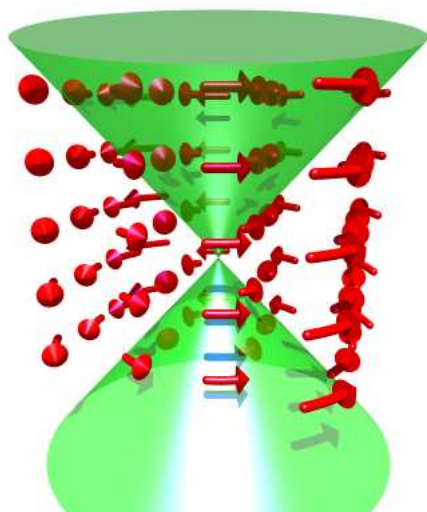
2b) (6 points) Match the following pictures with their vector fields and surfaces. Then check whether the flux integral is zero.



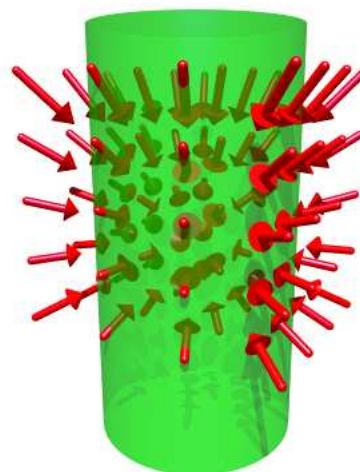
A



B



C

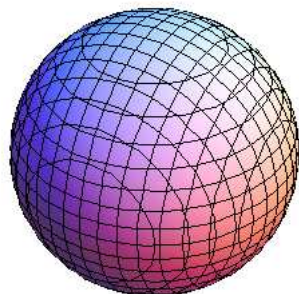


D

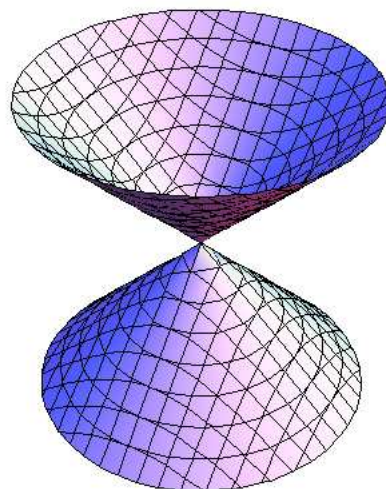
Enter A,B,C,D	Field	Surface	Flux zero
	$\vec{F}(x, y, z) = \langle x, y, z \rangle$	$\vec{r}(u, v) = \langle u, v, 0 \rangle$	
	$\vec{F}(x, y, z) = \langle 0, 0, y \rangle$	$\vec{r}(\theta, \phi) = \langle \sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), \cos(\phi) \rangle$	
	$\vec{F}(x, y, z) = \langle -x, -y, -z \rangle$	$\vec{r}(\theta, z) = \langle \cos(\theta), \sin(\theta), z \rangle$	
	$\vec{F}(x, y, z) = \langle -y, x, 0 \rangle$	$\vec{r}(\theta, z) = \langle z \cos(\theta), z \sin(\theta), z \rangle$	

Problem 3) (10 points)

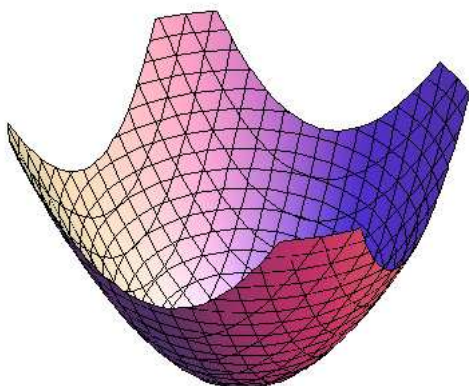
a) (6 points) Match the following level surfaces with functions $f(x, y, z)$ and also match the parametrization of part of the surface $f(x, y, z) = 0$.



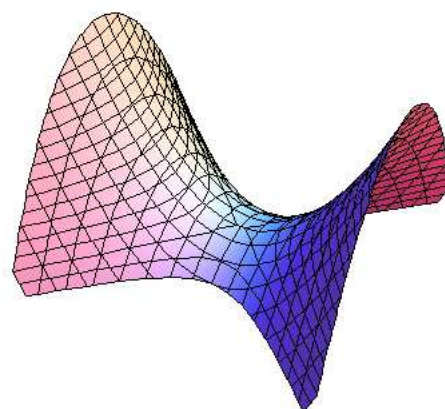
I



II



III



IV

Enter I,II,III,IV	$f(x, y, z) = 0$	Enter I,II,III,IV	parametrization
	$f(x, y, z) = -x^2 + y^2 + z$		$\langle u, v, u^2 - v^2 \rangle$
	$f(x, y, z) = x^2 + y^2 + z^2 - 1$		$\langle u, v, u^2 + v^2 \rangle$
	$f(x, y, z) = -x^2 - y^2 + z$		$\langle u, v, \sqrt{1 - u^2 - v^2} \rangle$
	$f(x, y, z) = -x^2 - y^2 + z^2$		$\langle s \cos(t), s \sin(t), s \rangle$

b) (2 points) We know that
 $\vec{r}''(t) = \langle -\cos(t), -\sin(t), 0 \rangle$,
 $\vec{r}(0) = \langle 2, 3, 4 \rangle$ and
 $\vec{r}'(0) = \langle 0, 1, 1 \rangle$.
 The expression $\langle \cos(t)+1, \sin(t)+3, t+4 \rangle$ is equal to:

Check which applies	result
☐	the velocity $\vec{r}'(t)$
☐	the position $\vec{r}(t)$
☐	the curvature $\kappa(\vec{r}(t))$
☐	the unit tangent vector $\vec{T}(t)$

c) (2 points) What is the name of the partial differential equation $\text{div}(\text{grad}(f)) = 0$ for $f(x, y)$?

Check which applies	PDE
☐	Transport equation
☐	Wave equation
☐	Heat equation
☐	Laplace equation

Problem 4) (10 points)

Find the distance between the sphere $(x-4)^2 + y^2 + (z-6)^2 = 1$ and the cylinder of radius 2 around the line $x = y = z$.

Problem 5) (10 points)

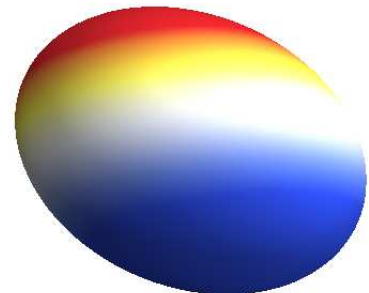
a) (3 points) Find the tangent plane to the surface $S : 4xy - z^2 = 0$ at $(1, 1, 2)$.

b) (4 points) Estimate $4 * 1.001 * 0.99 - 2.001^2$, where $*$ is the usual multiplication.

c) (3 points) Parametrize the line through $(1, 1, 2)$ which is perpendicular to the surface S at $(1, 1, 2)$.

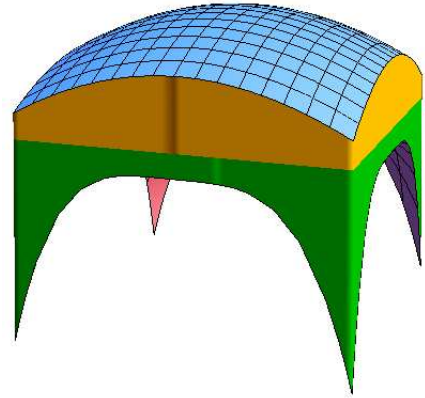
Problem 6) (10 points)

Find the place on the elliptical **asteroid** surface $g(x, y, z) = 5x^2 + y^2 + 3z^2 = 9$, where the temperature $f(x, y, z) = 750 + 5x - 2y + 9z$ is maximal.



Problem 7) (10 points)

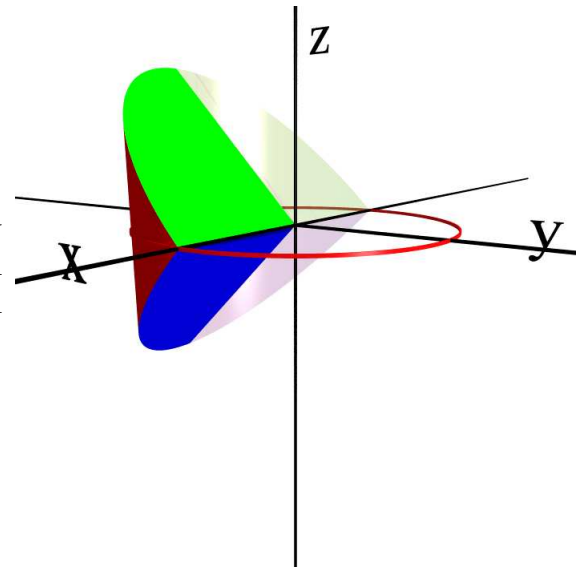
The thickness of the region enclosed by the two graphs $f_1(x, y) = 10 - 2x^2 - 2y^2$ and $f_2(x, y) = -x^4 - y^4 - 2$ is denoted by $f(x, y) = f_1(x, y) - f_2(x, y)$. Classify all critical points of f and find the global minimal thickness.



To the picture: over a square domain, the region looks like a chair. In the problem you consider the function over the entire plane.

Problem 8) (10 points)

Find the volume of the solid piece of **cheese** bound by the cylinder $x^2 + y^2 = 1$, the planes $y - z = 0$ (bottom boundary) and $y + z = 0$ (top boundary) which is on the quadrant $x \geq 0$ and $y \leq 0$.



Problem 9) (10 points)

Compute the surface area of the **Tsai** surface which is parametrized by

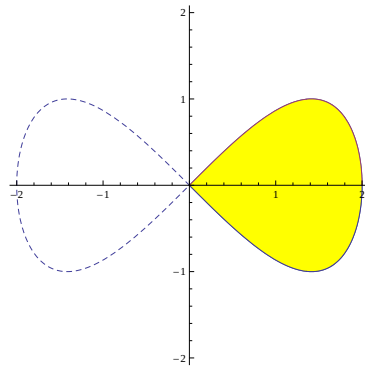
$$\vec{r}(u, v) = \langle 3u + 2v, 4u + v, \frac{2}{7}v^{\frac{7}{2}} \rangle,$$

where $0 \leq u \leq 1$ and $u^{1/4} \leq v \leq 1$.

Problem 10) (10 points)

Find the area $\int \int_R 1 \, dx dy$ of the region R inside the right leaf of the **Geronno lemniscate** $x^4 = 4(x^2 - y^2)$ which has the parametrization

$$\vec{r}(t) = \langle 2 \sin(t), 2 \sin(t) \cos(t) \rangle .$$



Problem 11) (10 points)

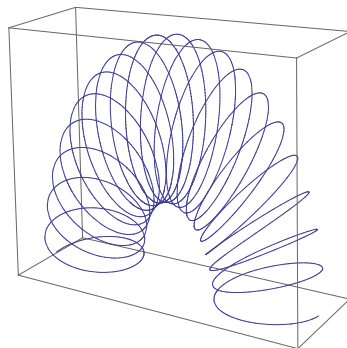
Find the line integral of the vector field

$$\vec{F}(x, y, z) = \langle \cos(x + z), 2yz e^{y^2 z} + 7, \cos(x + z) + y^2 e^{y^2 z} \rangle$$

along the **slinky** curve

$$\vec{r}(t) = \langle \sin(40t), (2 + \cos(40t)) \cos(t), (2 + \cos(40t)) \sin(t) \rangle$$

with $0 \leq t \leq \pi$.



Problem 12) (10 points)

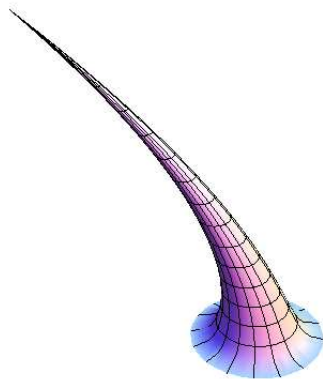
Find the flux integral $\int \int_S \text{curl}(\vec{F}) \cdot d\vec{S}$, where

$$\vec{F}(x, y, z) = \langle 2 \cos(\pi y) e^{2x} + z^2, x^2 \cos(z\pi/2) - \pi \sin(\pi y) e^{2x}, 2xz \rangle$$

and S is the **thorn** surface parametrized by

$$\vec{r}(s, t) = \langle (1 - s^{1/3}) \cos(t) - 4s^2, (1 - s^{1/3}) \sin(t), 5s \rangle$$

with $0 \leq t \leq 2\pi, 0 \leq s \leq 1$ and oriented so that the normal vectors point to the outside of the thorn.

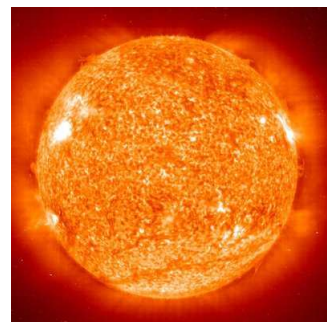


Problem 13) (10 points)

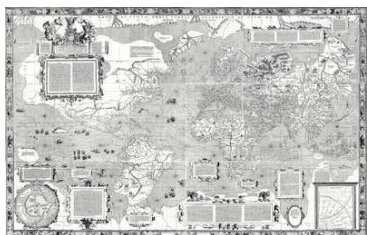
Assume the vector field

$$\vec{F}(x, y, z) = \langle 5x^3 + 12xy^2, y^3 + e^y \sin(z), 5z^3 + e^y \cos(z) \rangle$$

is the magnetic field of the **sun** whose surface is a sphere of radius 3 oriented with the outward orientation. Compute the magnetic flux $\int_S \vec{F} \cdot d\vec{S}$.



Problem 14) (10 points)



The **Mercator projection** is one of the most famous map projections. It was invented in 1569 and used for nautical voyages. The inverse of the projection is the parametrization of the sphere as

$$\vec{r}(u, v) = \langle \cos(u) \cos(\arctan(\sinh(v))), \sin(u) \cos(\arctan(\sinh(v))), \sin(\arctan(\sinh(v))) \rangle .$$

- (3 points) Show that $|\vec{r}(u, v)| = 1$ verifying so that $\vec{r}(u, v)$ parametrizes the unit sphere, if $0 \leq u < 2\pi, -\infty < v < \infty$.
- (3 points) Show that $|\vec{r}_u(u, v)| = |\vec{r}_v(u, v)| = 1/\cosh(v)$ and that $\vec{r}_u(u, v) \cdot \vec{r}_v(u, v) = 0$.
- (2 points) Use *b)* to show that $|\vec{r}_u \times \vec{r}_v| = 1/\cosh(x)^2$.
- (2 points) Use $\int 1/\cosh^2(x) dx = 2 \arctan(\tanh(x/2)) + C$ to see that the surface area of the unit sphere is 4π .

Hint for b): you can use the identity $\cos(\arctan(\sinh(v))) = 1/\cosh(v)$.