11/7/2013 SECOND HOURLY **Math 21a, Fall 2013**

Name:

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MWF 10 Gijs Heuts
MWF 10 Adrian Zahariuc
MWF 10 Yihang Zhu
MWF 11 Peter Garfield
MWF 11 Matthew Woolf
MWF 12 Charmaine Sia
MWF 12 Steve Wang
MWF 14 Mike Hopkins
TTH 10 Oliver Knill
TTH 10 Francesco Cavazzani
TTH 11:30 Kate Penner
TTH 11:30 Francesco Cavazzani

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) True/False questions (20 points), no justifications needed

1) T F — For any continuous function $f(x,y)$, we have $\int_0^1 \int_1^2 f(x,y)\,dx\,dy = \int_1^2 \int_0^1 f(x,y)\,dx\,dy$.

Solution:
This is not Fubini

2) T F — If $\vec{u}$ is a unit vector tangent to $f(x,y) = 1$ at $(0,0)$ and $f(0,0) = 1$, then $D_{\vec{u}} f(0,0)$ is zero.

3) T F — Assume f is zero on $x = y$ and $x = -y$, then $(0,0)$ is a critical point of f.

4) T F — If $(0,0)$ is the only local minimum of a function f and f has no local maxima, then $(0,0)$ is a global minimum.

5) T F — If $(0,0)$ is a critical point for f, and $f_{yy}(0,0) < 0$ then $(0,0)$ is not a local minimum.

6) T F — If $f(x,y)$ and $g(x,y)$ have the same non-constant linearization $L(x,y)$ at $(0,0)$ and $f(0,0) = g(0,0) = 0$, then the level sets $f = 0$ and $g = 0$ have the same tangent line at $(0,0)$.

7) T F — There are saddle points with positive discriminant $D > 0$.

Solution:
No. By definition not

8) T F — If R is the unit disc, then $\int\int_R x^2 - y^2 \; dxdy$ is zero.

Solution:
Yes. This can be seen as the definition of area.

9) T F — There is a nonzero function $f(x,y)$ for which the linearization $L(x,y)$ is equal to $2f(x,y)$.

Solution:
Yes a linear function

10) T F — The directional derivative at a local minimum $(0,0)$ is positive in every direction.

Solution:
It is zero

11) T F If $\vec{r}(t)$ is a curve on the surface $g(x, y, z) = 1$, then $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

Solution:
Use the chain rule and the fact that $g(\vec{_(}t))$ is constant so that $d/dtg(\vec{_(}t))$ is zero.

12) T F If $|\nabla f(0, 0)| = 2$, there is a direction in which the directional derivative at $(0, 0)$ is 2.

Solution:
Yes, we have seen this computation

13) T F If $D > 0$ at $(0, 0)$ and $\nabla f(0, 0) = 0$ and $f_{xx}(0, 0) < 0$ then $f_{yy}(0, 0) < 0$.

Solution:
We have seen this from $D = f_{xx}f_{yy} - f_{xy}^2$.

14) T F $\int_0^1 \int_0^x f(x, y)\, dydx = \int_0^1 \int_y^1 f(x, y)\, dxdy$.

Solution:
This is almost a correct correct switch but the $dxdy$ have also to be switched.

15) T F The surface area of the sphere of radius L is $\int_0^\pi L^2 \sin(\phi)\, d\phi$.

Solution:
The answer is 4π.

16) T F If $f(x, y) = g(x)$ is a function of x only, then $D = 0$ at every critical point.

Solution:
Indeed, then $f_{xy}, f_{xx} f_{yy}$ are both zero.

17) | T | **F** | The gradient vector $\nabla f(x_0, y_0)$ is a vector which is perpendicular to the surface $z = f(x, y)$.

Solution:
Nonsense

18) | **T** | F | If $|\nabla f(0,0)| = 2$, then there is a unit vector $\vec{v}$ such that $D_{\vec{v}} f(0,0) = 1$.

Solution:
Intermediate value theorem.

19) | **T** | F | The gradient of the function $f(x, y) = \int_x^y \sin(t)\, dt$ is $\langle -\sin(x), \sin(y) \rangle$.

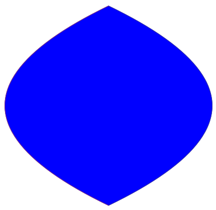
Solution:
Yes, this is the derivative by the fundamental theorem.

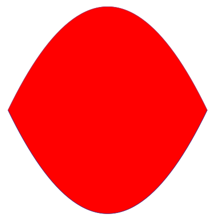
20) | **T** | F | Assume $f(x, y) = x^2 + y^4$ and a curve $\vec{r}(t)$ satisfies $\vec{r}\,'(t) = \nabla f(\vec{r}(t))$, then $\frac{d}{dt} f(\vec{r}(t)) \geq 0$.

Solution:
FTTFTTFTFFTTTTFTFTTT

Problem 2) (10 points) No justifications needed

a) (6 points) Match the regions with the integrals. Each integral matches one region $A-F$.


A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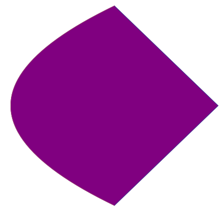
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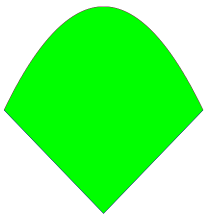
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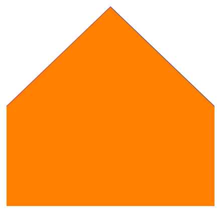
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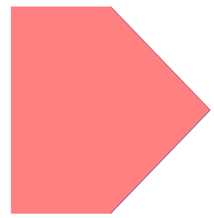

F

Enter A-F	Integral
	$\int_{-1}^{1} \int_{-1}^{2-\|y\|} f(x,y)\ dxdy$
	$\int_{-1}^{1} \int_{y^2}^{2-\|y\|} f(x,y)\ dxdy$
	$\int_{-1}^{1} \int_{x^2}^{2-x^2} f(x,y)\ dydx$
	$\int_{-1}^{1} \int_{\|x\|}^{2-x^2} f(x,y)\ dydx$
	$\int_{-1}^{1} \int_{y^2}^{2-y^2} f(x,y)\ dxdy$
	$\int_{-1}^{1} \int_{-1}^{2-\|x\|} f(x,y)\ dydx$

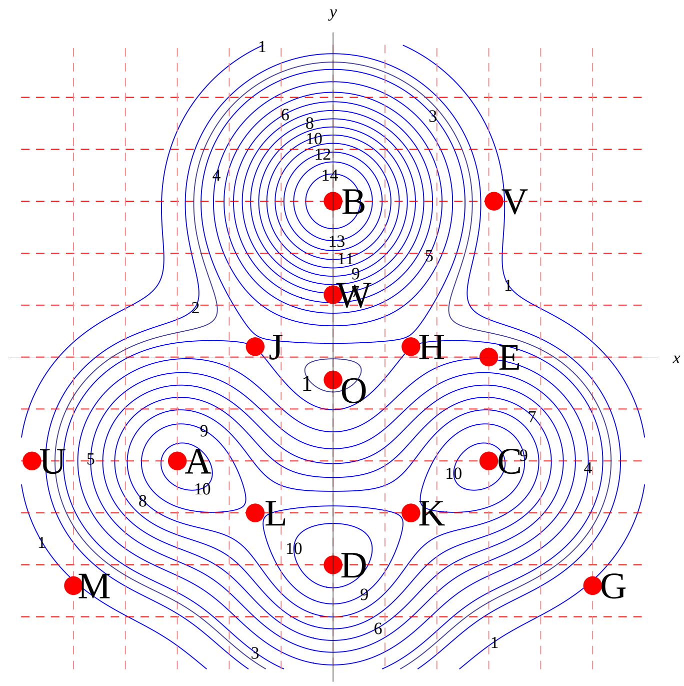
b) (4 points) Name the partial differential equations correctly. Each equation matches one name.

Fill in 1-4	Name
	Laplace
	Wave
	Transport
	Heat

Equation Number	PDE
1	$g_x - g_y = 0$
2	$g_{xx} - g_{yy} = 0$
3	$g_x - g_{yy} = 0$
4	$g_{xx} + g_{yy} = 0$

Solution:
FCBDAE
4,2,1,3

Problem 3) (10 points)



a) (6 points) Enter one label into each of the boxes.

At which point is the length of the gradient maximal?

At which point is the global maximum?

At which point is $f_x > 0, f_y = 0$?

At which point is $D_{\langle 1,1\rangle/\sqrt{2}} f = 0$, $D_{\langle 1,-1\rangle/\sqrt{2}} f < 0$?

At which point is f maximal under the constraint $g(x, y) = y = 0$?

At which point does f have a local minimum?

Solution:

b) (4 points) Note that the zero vector is considered both parallel and perpendicular to any other vector.

	parallel	perp	
The gradient ∇f is always			to the surface $f = c$.
For a Lagrange minimum, ∇g is			to ∇f.
If $(0,0)$ is a min. of f then $\nabla f(0,0)$ is			to $\langle 1, 0 \rangle$.
If $(0,0)$ is max. of f and $g = z - f(x,y)$ then ∇g is			to $\langle 0, 0, 1 \rangle$.

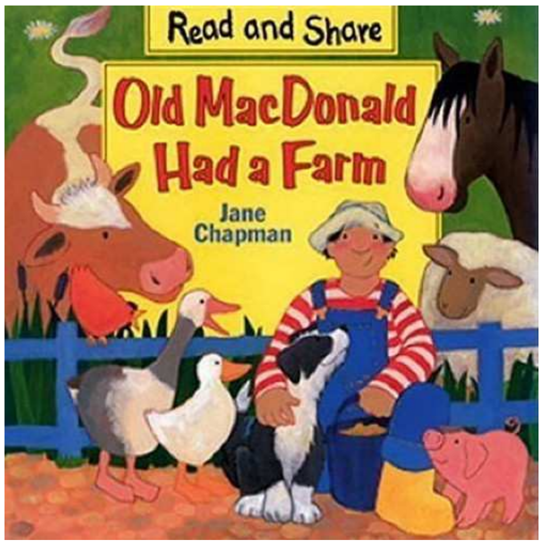
Solution:
WBUGEO

	parallel	perp	
The gradient ∇f is always		*	to the surface $f = c$.
For a Lagrange minimum, ∇g is	*		to ∇f.
If $(0,0)$ is a min. of f then $\nabla f(0,0)$ is	*	*	to $\langle 1, 0 \rangle$.
If $(0,0)$ is max. of f and $g = z - f(x,y)$ then ∇g is	*		to $\langle 0, 0, 1 \rangle$.

Problem 4) (10 points)

A farm costs $f(x,y)$, where x is the number of cows and y is the number of ducks. There are 10 cows and 20 ducks and $f(10,20) = 1000000$. We know that $f_x(x,y) = 2x$ and $f_y(x,y) = y^2$ for all x, y. Estimate $f(12,19)$.

"Old MacDonald had a million dollar farm, E-I-E-I-O,
and on that farm he had $x = 10$ cows, E-I-E-I-O,
and on that farm he had $y = 20$ ducks, E-I-E-I-O,
with $f_x = 2x$ here and $f_y = y^2$ there,
and here two cows more, and there a duck less,
how much does the farm cost now, E-I-E-I-O?"



Solution:
$f(10,20) = 1000000$. The linearization is $L(12,19) = f(10,20) + 2(10)(12-10) + (20)^2(19-20) = 1000000 + 2(20) - 400 = 999640$.

Problem 5) (10 points)

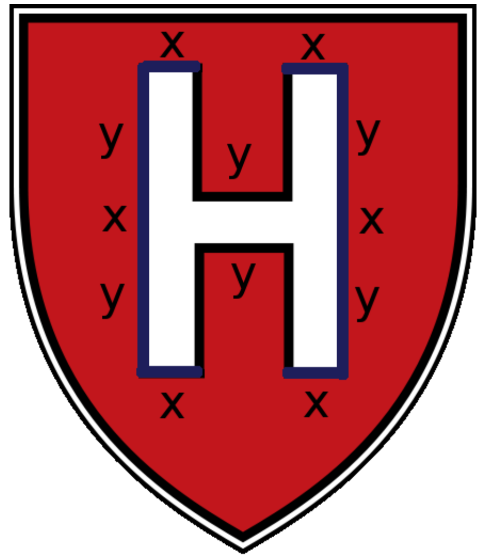
Find the Harvard H which has maximal area

$$f(x,y) = 5xy + 2x^2$$

with fixed exposed perimeter

$$6x + 4y = 88 \ .$$

Find the maximum using Lagrange.



Solution:
The Lagrange equations are

$$\begin{aligned} 5y + 4x &= 6\lambda \\ 5x &= 4\lambda \\ 6x + 4y &= 88 \end{aligned}$$

Eliminating λ gives $y = (7/10)x$. $x = 10, y = 7, f(10,7) = 550$.

Problem 6) (10 points)

a) (7 points) A minigolf on the cape has a hole at a local minimum of the function

$$f(x,y) = 3x^2 + 2x^3 + 2y^5 - 5y^2 \ .$$

Find all the critical points and classify them.

b) (3 points) A golfer hits tangent to the level curve $f(x,y) = 2$ through $(1,1)$. Find this line.

About minigolf: the first standardized minigolf course appeared in 1916 in North Carolina. The world record on a round of minigolf is 18 strokes on 18 holes on eternite. No perfect round on concrete has been scored. The highest prizes reach 5000 dollars only so that nobody is known to make a living by competing in minigolf.

Solution:

a)

-1	0	60	-6	min	1
-1	1	-180	-6	saddle	-2
0	0	-60	6	saddle	0
0	1	180	6	min	-3

b) We know the gradient at the point. $x = 1$.

Problem 7) (10 points)

A circular track near Salem is a circle of radius 500 which is centered at the origin $(0, 0)$. A go-kart goes counter-clockwise around the track $\vec{r}(t)$. The cheering intensity is given by a function $f(x, y)$. The go-kart passes the point $(300, 400)$ at time $t = 0$ with velocity $\langle -4, 3 \rangle$. We know that $f_x(300, 400) = 2$ and $f_y(300, 400) = 10$. Find the rate of change

$$\frac{d}{dt} f(\vec{r}(t))$$

at $t = 0$.

Solution:
Use the chain rule:

$$\frac{d}{dt} f(r(t)) = \nabla f(r(t)) \cdot r'(t)$$

At $t = 0$, we have $\langle 2, 10 \rangle \cdot \langle -4, 3 \rangle = -8 + 30 = 22$.

Problem 8) (10 points)

a) (6 points) Find the integral

$$\int_0^1 \int_y^{y^{1/5}} \frac{e^x + x^7}{x - x^5} \, dx \, dy \; .$$

b) (4 points) Integrate

$$\int_{-1}^0 \int_0^{\sqrt{1-y^2}} \frac{e^{\sqrt{x^2+y^2}}}{\sqrt{x^2 + y^2}} \, dxdy \; .$$

Solution:
a) Change the order of integration $e - 7/8$
b) Use polar coordinates $\pi/2(e-1)$. Note that the region is in the fourth quadrant so that we integrate from $-\pi/2$ to 0.

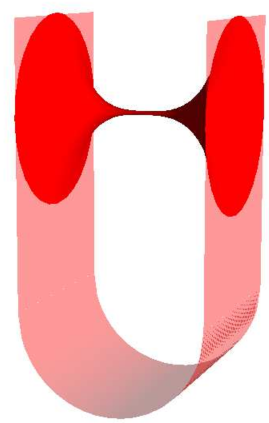
Problem 9) (10 points)

Find the surface area of the "wormhole"

$$\vec{r}(u,v) = \langle 3v^3, v^9\cos(u), v^9\sin(u)\rangle \ ,$$

where $0 \le u \le 2\pi$ and $-1 \le v \le 1$.

Einstein-Rosen bridges *are hypothetical topological constructions which would allow shortcuts through spacetime. Tunnels connecting different parts of the universe appear frequently in science fiction.*



Solution:
Note that $\int\int |\vec{r}_u \times \vec{r}_v| \; dudv$ has a nonnegative integrand. In our case it is $9v^{11}\sqrt{1+v^{12}}$. When we integrate this from 0 to 2π we get $18\pi v^{11}\sqrt{1+v^{12}}$. But note that we have to take either square root so that the integrand is nonnegative or just take the absolute value. In any case, we can just take twice the integral from 0 to 1. The answer is not zero but $2\pi(\sqrt{8}-1)$.