11/7/2013 SECOND HOURLY PRACTICE III **Math 21a, Fall 2013**

Name:

MWF 9 Oliver Knill
MWF 9 Chao Li
MWF 10 Gijs Heuts
MWF 10 Adrian Zahariuc
MWF 10 Yihang Zhu
MWF 11 Peter Garfield
MWF 11 Matthew Woolf
MWF 12 Charmaine Sia
MWF 12 Steve Wang
MWF 14 Mike Hopkins
TTH 10 Oliver Knill
TTH 10 Francesco Cavazzani
TTH 11:30 Kate Penner
TTH 11:30 Francesco Cavazzani

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1) T **F** If $f(x, y) = 1$ is a curve, and near $(2, 3)$ one can write y as a function of x, then $y' = -f_y(2, 3)/f_x(2, 3)$.

Solution:
The order is wrong: the f_y is in the denominiator.

2) T **F** If $\iint_R f(x, y)\, dA = 0$, then the function $f(x, y)$ is everywhere zero on $R = \{x^2 + y^2 \leq 1\}$.

Solution:
If $f = xy$ and R is the disc, then the integral is zero but f is nonzero.

3) **T** F The directional derivative in the direction of the gradient is $|\nabla f|$.

Solution:
Indeed, $D_{\nabla f/|\nabla f|} = \nabla f \cdot \nabla f/|\nabla f| = |\nabla f|$.

4) T **F** The linearization of $f(x, y) = x^3 + y^3$ at $(1, 1)$ is the quadratic function $L(x, y) = 3x^2 + 3y^2$.

Solution:
The linearization is a linear (affine) function and not quadratic.

5) T **F** The function $f(x, y) = x^2 + y^2$ satisfies the partial differential equation $D = f_{xx}f_{yy} - f_{xy}^2 = 1$.

Solution:
Almost, a factor 4 is missing.

6) T **F** The function x^2y^2 has no local minimum at $(0, 0)$ because the discriminant function D is zero there.

Solution:
There can be a local minimum with $D = 0$.

7) T F

The double integral $\int_0^{\pi/4} \int_0^2 r^3 \; dr d\theta$ is the volume of the part of a solid cylinder $x^2 + y^2 \leq 4$ which is below the paraboloid $z = x^2 + y^2$ and above the xy plane.

Solution:
We do not integrate from 0 to 2π.

8) T F

The gradient of $f(x, y, z)$ at (x_0, y_0, z_0) is perpendicular to the level surface of f through (x_0, y_0, z_0).

Solution:
It indeed is! This is an important fact.

9) T F

If $f(x, y, z) = 3x - 4z$, then the minimal possible directional derivative $D_{\vec{u}} f$ at any point in space is -5.

Solution:
The gradient has length 5. The directional derivative into the direction of the gradient is the length of the gradient.

10) T F

If (x, y) is not a critical point, then the directional derivative $D_{\vec{v}} f$ can take both positive and negative values for different choices of $\vec{v}$.

Solution:
The directional derivative changes sign if $\vec{v}$ is replaced by $-\vec{v}$.

Solution:
Because of the symmetry $D_{-v} f = -D_v f$ the integral is zero.

11) T F

Using linearization of $f(x, y) = x/y$ we can estimate $1.01/1.001 = f(1.01, 1.001) \sim 1 + 0.01 - 0.001 = 1.009$.

Solution:
$L(x, y) = 1 + 1 \cdot 0.01 - 1 \cdot 0.001$.

12) | T | **F** | If $(0,0)$ is a critical point of $f(x,y)$ with nonzero discriminant $D = f_{xx}f_{yy} - f_{xy}^2$, we know that it is either a saddle, a global maximum or a global minimum.

Solution:
Local max or min, but not necessary global max or min.

13) | T | **F** | For a rectangular region R, Fubini tells that $\int_0^2 \int_0^3 f(x,y) \ dxdy = \int_0^2 \int_0^3 f(x,y)\ dydx$ for any continuous function $f(x,y)$.

Solution:
We also have to switch the integration bounds.

14) | T | **F** | If a function $f(x,y)$ has only one critical point $(0,0)$ in $G = \{x^2 + y^2 \leq 1\}$ which is a local maximum and $f(0,0) = 1$, then $\iint_G f(x,y)\ dxdy > 0$.

Solution:
The critical point can be surrounded by a small region only, where f is positive.

15) | **T** | F | If $\vec{r}(t)$ is a curve in space for which the speed is 1 at all times and $f(x,y,z)$ is a function of three variables, then $d/dt f(\vec{r}(t)) = D_{\vec{r}\,'(t)}(f)$.

Solution:
Yes, this is the chain rule.

16) | **T** | F | $\int_0^1 \int_0^1 f_{xy}(x,y)\ dydx = f(1,1) - f(1,0) - f(0,1) + f(0,0)$.

Solution:
This is a consequence of the fundamental theorem of calculus

17) | **T** | F | If $f_{yy}(x,y) > 0$ everywhere, then f can not have any local maximum.

Solution:
We would have $f_{yy} < 0$ at a local maximum.

18) T F

The double integral $\int_0^1 \int_0^1 x^2 - y^2 \, dxdy$ is the volume of the solid below the graph of $f(x, y) = x^2 - y^2$ and above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ in the xy-plane.

Solution:
It is a signed volume. There can be part below.

19) T F

For any unit vector $\vec{v}$ and any differentiable function f, one has $D_{\vec{v}}(f) + D_{-\vec{v}}(f) = 0$.

Solution:
Write down the definition. The sum is $\nabla f \cdot (v - v) = 0$.

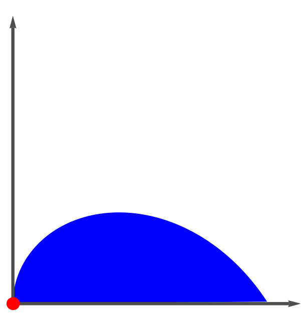
20) T F

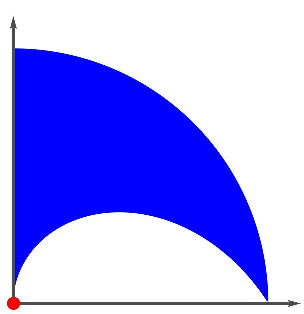
The surfaces $x + y + z = 0$ and $x^2 + y^2 + z^2 + x + y + z = 0$ have the same tangent plane at $(0, 0, 0)$.

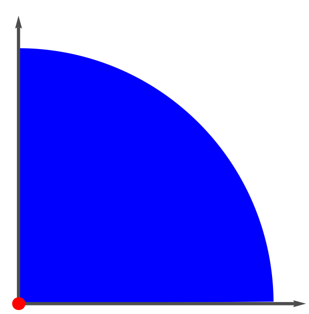
Solution:
They have the same gradient at $(0, 0, 0)$.

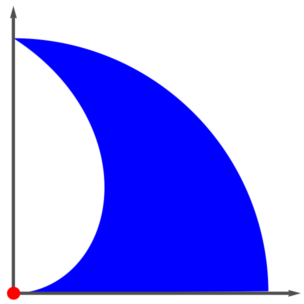
Problem 2) (10 points)

a) (6 points) Match the regions with the corresponding polar double integrals

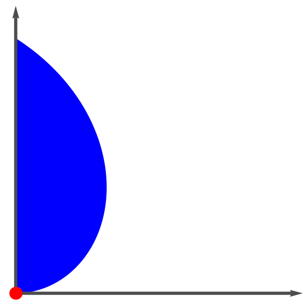

A


B

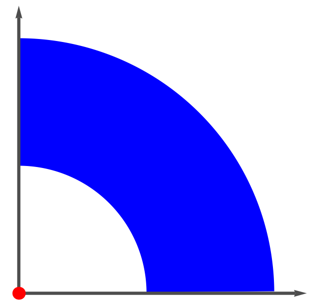

C



D



E



F

Enter A-F	Integral of $f(r,\theta)$
	$\int_0^{\pi/2}\int_0^{\pi/2} f(r,\theta) r\ dr d\theta$
	$\int_0^{\pi/2}\int_0^{\theta} f(r,\theta) r\ dr d\theta$
	$\int_0^{\pi/2}\int_0^{\pi/2-\theta} f(r,\theta) r\ dr d\theta$

Enter A-F	Integral of $f(r,\theta)$
	$\int_0^{\pi/2}\int_{\theta}^{\pi/2} f(r,\theta) r\ dr d\theta$
	$\int_0^{\pi/2}\int_{\pi/2-\theta}^{\pi/2} f(r,\theta) r\ dr d\theta$
	$\int_0^{\pi/2}\int_{\pi/4}^{\pi/2} f(r,\theta) r\ dr d\theta$

b) (4 points) Match the partial differential equations (PDE's) for the functions $u(t,s)$ with their names. No justifications are needed.

Enter A,B,C,D here	PDE
	$u_t + u u_s - u_{ss} = 0$
	$u_{tt} + u_{ss} = 0$

Enter A,B,C,D here	PDE
	$u_{tt} - u_{ss} = 0$
	$u_t - u_{ss} = 0$

A) Wave equation	B) Heat equation	C) Burgers equation	D) Laplace equation

Solution:

	C	D
a)	E	B
	A	F
b)	C	A
	D	B

Problem 3) (10 points)

a) (7 points) Find and classify all the critical points of the function

$$f(x,y) = 5 + 3x^2 + 3y^2 + y^3 + x^3 \ .$$

b) (3 points) Is there a global maximum or a global minimum for $f(x,y)$?

Solution:
The gradient of f is

$$\langle 6x + 3x^2, 6y + 3y^2 \rangle = \langle 3x(2+x), 3y(2+y) \rangle \ .$$

There are 5 critical points:

x	y	D	f_{xx}	Type	f value
-2	-2	36	-6	maximum	13
-2	0	-36	-6	saddle	9
0	-2	-36	6	saddle	9
0	0	36	6	minimum	5

b) There is no global maximum (nor global minimum). For $y = 0$, have $5 + 3x^2 + x^3$ which grows like x^3 for $x \to \infty$.

Problem 4) (10 points)

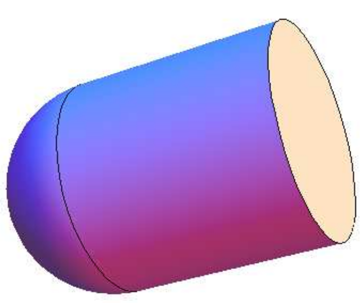
A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi r h + \pi r^2$. Doctor Manhatten designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes

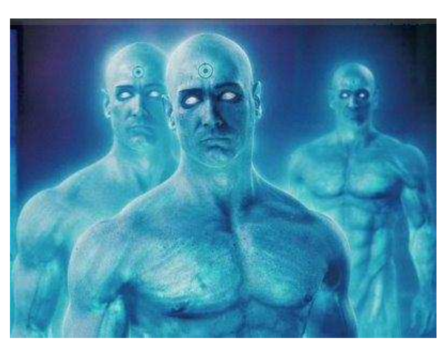
$$f(h,r) = 3r^2 + 2rh$$

under the constraint

$$g(h,r) = 2r^3 + 3r^2 h = 1 \ .$$

Use the Lagrange method to find a local minimum of f under the constraint $g = 1$.

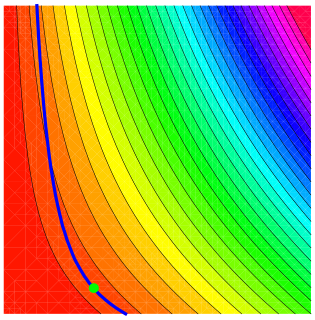




Solution:
The Lagrange equations are

$$\begin{aligned} 2h + 6r &= \lambda(6hr + 6r^2) \\ 2r &= 3\lambda r^2 \\ 3hr^2 + 2r^3 &= 1 \ . \end{aligned}$$

Because $r = 0$ is incompatible with the third equation, we can divide the second equation by r. This allows to eliminate λ and $2h+6r = 4h+4r$ which is $h = r$. The third equation gives us $h = r = 1/5^{1/3}$. The point where the minimum occurs is $(1/5^{1/3}, 1/5^{1/3})$. The minimal value of f is $5/5^{2/3}$.

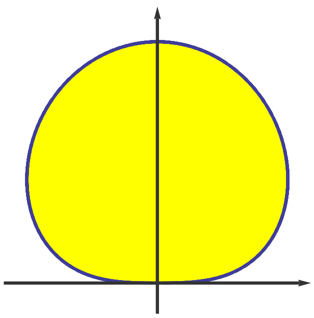


Problem 5) (10 points)

A region R in the plane shown to the right is called the **"blob of nothingness"**. It does not have any purpose nor meaning. It just sits there. The region is given in polar coordinates as $0 \leq r \leq \theta(\pi - \theta)$ for $0 \leq \theta \leq \pi$. Find the area

$$\int \int_R 1 \ dxdy$$

of this nihilistic object.



Solution:

$$\int_0^{\pi} \int_0^{\theta(\pi-\theta)} r \; dr d\theta = \int_0^{\pi} \theta^2(\pi-\theta)^2/2 \; d\theta = \pi^5/60 \; .$$

Problem 6) (10 points)

a) (4 points) If

$$f(x,y) = y\cos(x-y) \; ,$$

find equation of plane tangent to $z = f(x,y)$ at the point $(2,2,2)$.

b) (3 points) Find the equation of the tangent line to $f(x,y) = 2$ at $(2,2)$.

c) (3 points) Estimate $f(2.1, 1.9)$ using linear approximation.

Solution:
a) Define $g(x,y,z) = f(x,y) - z$. It is important to deal with a function of three variables when looking at planes. The graph of g is equal to the level surface $z - f(x,y) = 0$. Then $\nabla g(2,2,2) = \langle 0,1,-1 \rangle$. The tangent plane is of the form $y - z = d$ where d is a constant. Plugging in $(2,2,2)$ gives $y - z = 0$.
b) $\nabla f(x,y) = \langle -y\sin(x-y), \cos(x-y) + \sin(x-y)$ and $\nabla f(2,2) = \langle 0,2 \rangle$ so that the equation of the tangent line is $y = d$ for a constant d. Plugging in the point $(2,2)$ gives $y = 2$.
c) $L(2.1, 1.9) = 2 + 0(0.1) + 1(-0.1) = 1.9$.

Problem 7) (10 points)

A **Harvard robot bee** flies along the curve

$$\vec{r}(t) = \langle t - t^3, 3t^2 - 3t \rangle$$

and measures the temperature $f(x, y)$. It flies over the target point $(0, 0)$ at time $t = 0$ and time $t = 1$. At each time, its sensor measures the temperature change $g'(t)$ where $g(t) = f(\vec{r}(t))$.

Image source: Harvard Press release on robobees.seas.harvard.edu

a) (5 points) Assume you knew that the gradient of f at $(0, 0)$ is $\langle a, b \rangle$. What are the values of $g'(t) = d/dt f(\vec{r}(t))$ at $t = 0$ and $t = 1$ in terms of a and b?

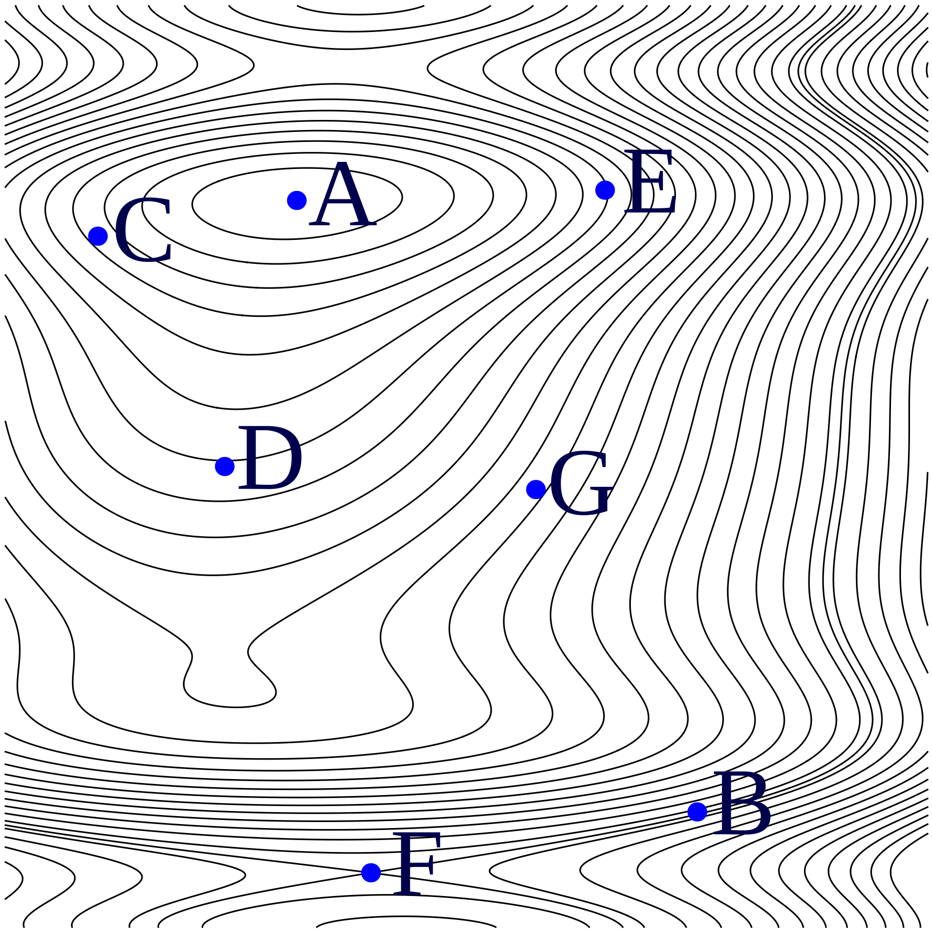
b) (5 points) The bee measures $g'(0) = 3$ and $g'(1) = 3$. What is the gradient $\nabla f(0, 0) = \langle a, b \rangle$ of f at $(0, 0)$?

Solution:
a) $\vec{r}'(t) = \langle 1 - 3t^2, 6t - 3 \rangle$. So that $\vec{r}'(0) = \langle 1, -3 \rangle, \vec{r}'(1) = \langle -2, 3 \rangle$. If the gradient of f at $(0, 0)$ is $\langle a, b \rangle$ we get by the chain rule $d/dt f(\vec{r}(t)) = \langle a, b \rangle \cdot \vec{r}\,'(t)$ which is either $\langle a, b \rangle \cdot \langle 1, -3 \rangle = a - 3b$ or $\langle a, b \rangle \cdot \langle -2, 3 \rangle = -2a + 3b$.
b) We know that $a - 3b = 3$ and $-2a + 3b = 3$. This is a system of linear equations which has the solution $\langle a, b \rangle = \langle -6, -3 \rangle$.

Problem 8) (10 points)

A function $f(x, y)$ of two variables has level curves as shown in the picture. The function values at neighboring level curves differ by 1. [No justifications are needed in this problem. Naturally, since there are less points then boxes, some of the points A-G will appear more than once, but each box will only be filled with one letter.]

Enter A-G	is a point, where ...
	$f_x(x, y) = 0$ and $f_y(x, y) \neq 0$.
	$f_y(x, y) = 0$ and $f_x(x, y) \neq 0$.
	$f(x, y)$ has either a max or a min.
	$f(x, y)$ has a saddle point.
	$f(x, y)$ has no max nor min but is extremal under a constraint $y = c$ for some c.
	$f(x, y)$ has no max nor min but is extremal under a constraint $x = c$ for some c.
	the length of the gradient vector of f is largest among all points A-G.
	$D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f(x, y) = 0$ and $D_{\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle} f(x, y) \neq 0$.
	$D_{\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle} f(x, y) = 0$ and $D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f(x, y) \neq 0$.
	the tangent line to the curve is $x + y = d$ for some constant d.



Solution:
D,E,A,F,(D or F), (E or F), B,G,C,C.
To the tangent line (last choice): The equation $x+y=d$ means that the gradient vector is $\langle 1,1 \rangle$ at the point. Use now that the gradient vector is perpendicular to the level curve.

Problem 9) (10 points)

Evaluate the following double integral

$$\int_0^1 \int_0^{(1-x)^2} \frac{x^3}{(1-\sqrt{y})^4} \, dydx \; .$$

Solution:
We have to change the order of integration

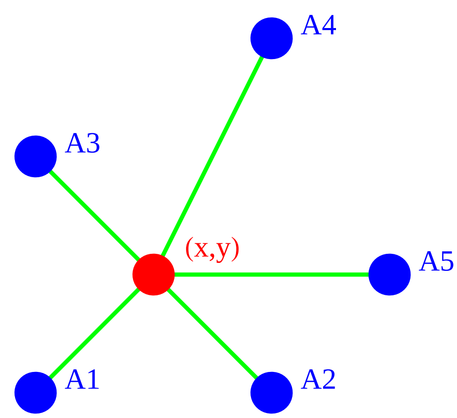
$$\int_0^1 \int_0^{1-\sqrt{y}} x^3/(1-\sqrt{y})^4 dxdy = \int_0^1 (1/4)\, dy = 1/4 \,.$$

Problem 10) (10 points)

A mass point with position (x, y) is attached by springs to the points $A_1 = (0, 0), A_2 = (2, 0), A_3 = (0, 2), A_4 = (2, 3), A_5 = (3, 1)$. It has the potential energy

$$f(x, y) = 31 - 14x + 5x^2 - 12y + 5y^2$$

which is the sum of the squares of the distances from (x, y) to the 5 points. Find all extrema of f using the second derivative test. The minimum of f is the position, where the mass point has the lowest energy.



Solution:
The gradient of f is

$$\nabla f(x, y) = \langle -14 + 10x, -12 + 10y \rangle \,.$$

It leads to the solution $(x, y) = (7, 6)/5 = (1.4, 1.2)$.
(Side remark: In general the average $\sum_{i=1}^n A_i/n$ is the only critical point because the function $f(X) = \sum_{i=1}^n |x - A_i|^2$ has the gradient $\sum_{i=1}^n 2(X - A_i) = 0$ showing $nX = \sum_i A_i$. This is true in any dimension and any number of mass points.)
The Hessian matrix is

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \,.$$

From this we can read $D = 100$ and $f_{xx} = 10$. The second derivative test shows that the point is a minimum. We have $f(1.4, 1.2) = 14$.