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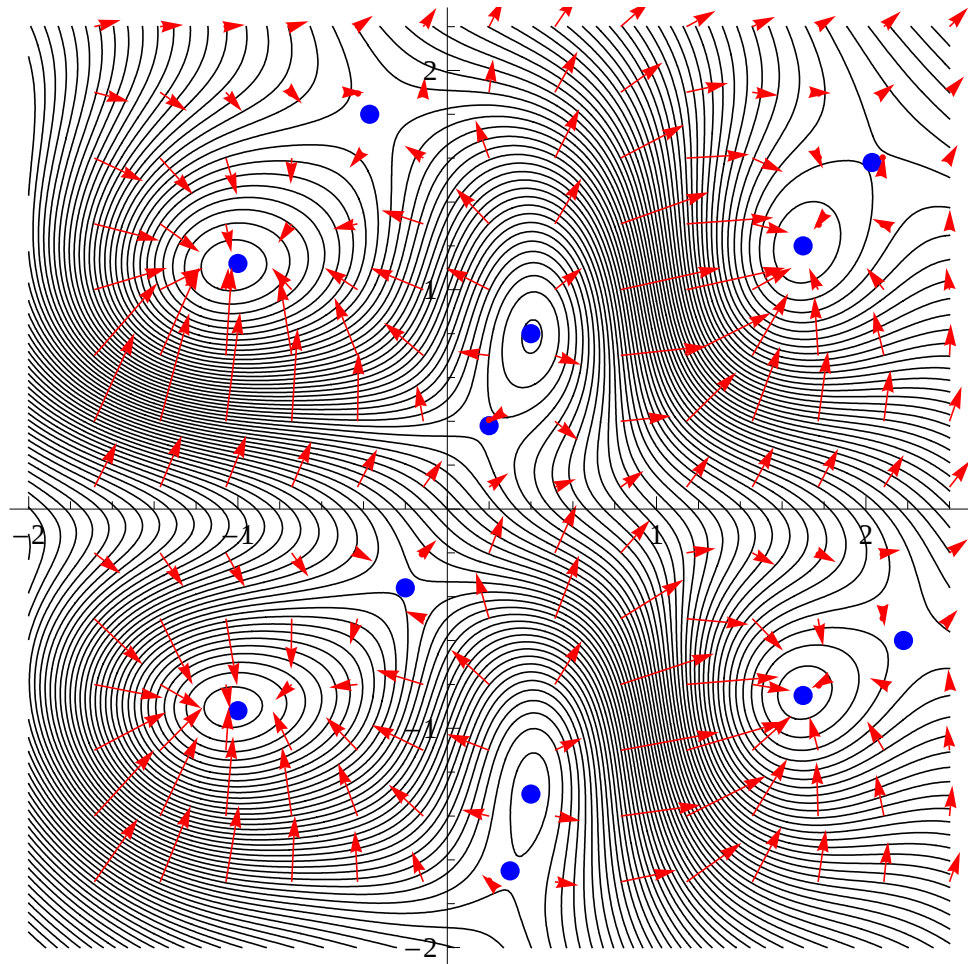
- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

- | | | |
|-----|---|--|
| 1) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | Given a unit vector v , define $g(x) = D_v f(x)$. If at a critical point, for all vectors v we have $D_v g(x) > 0$, then f is a local maximum. |
| 2) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | Assume f satisfies the PDE $f_x = f_y$. If $g = f_x$, then $g_x = g_y$. |
| 3) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | The equation $\phi = \pi/4$ in spherical coordinates ($\rho \geq 0, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi$ as usual) and the surface $x^2 + y^2 = z^2$ (with no further restrictions on x, y, z) are the same surface. |
| 4) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | Even with $f_x(a, b) = 0$ and $f_y(a, b) = 0$, it is possible that some directional derivative $D_{\vec{v}}(f)$ of $f(x, y)$ at (a, b) is non-zero. |
| 5) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | There exists a pair of different points on a sphere, for which the tangent planes are parallel. |
| 6) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | If $\vec{u}$ is a unit vector tangent at (x, y, z) to the level surface of $f(x, y, z)$ then $D_{\vec{u}} f(x, y, z) = 0$. |
| 7) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | Assume we have a smooth function $f(x, y)$ for which the lines $x = 0, y = 0$ and $x = y$ are level curves $f(x, y) = 0$. Then $(0, 0)$ is a critical point with $D < 0$. |
| 8) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | The gradient of $f(x, y)$ is perpendicular to the graph of f . |
| 9) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | The level curves of a linearization $L(x, y)$ of a function $f(x, y) = \sin(x + y)$ at $(0, 0)$ consist of lines. |
| 10) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | If $x^4 y + \sin(y) = 0$ then $y' = 4x^3 y / (x^4 + \cos(y))$. |
| 11) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | The linearization $L(x, y)$ at a critical point (x_0, y_0) of a function $f(x, y)$ is a constant function. |
| 12) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | The surface $x^2 + y^2 - z^2 = 1$ has a parametrization of the form $\langle x(s, t), y(s, t), z(s, t) \rangle = \langle s, t, f(s, t) \rangle$ for some function $f(s, t)$ for which the parametrization covers the entire surface. |
| 13) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | The tangent plane to the graph of $f(x, y)$ at a point $(x_0, y_0, f(x_0, y_0))$ is a level surface of the linearization $L(x, y, z)$ of $z - f(x, y)$. |
| 14) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | The critical points of $F(x, y, \lambda) = f(x, y) - \lambda g(x, y)$ are solutions to the Lagrange equations when extremizing the function $f(x, y)$ under the constraint $g(x, y) = 0$. |
| 15) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | The curve defined by $z = 1, \theta = \frac{\pi}{4}$ in cylindrical coordinates is a circle. |
| 16) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | If $(0, 0)$ is a critical point of $f(x, y)$ and the discriminant D is zero but $f_{xx}(0, 0) > 0$ then $(0, 0)$ can not be a local maximum. |
| 17) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | If $f(x, y, z) = x^2 + y^2 + z^2$, then $\nabla f = 2x + 2y + 2z$. |
| 18) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | A function $f(x, y)$ in the plane always has a local minimum or a local maximum. |
| 19) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | For any smooth function $f(x, y)$, the inequality $\ \nabla f\ \geq f_x + f_y $ is true. |
| 20) | <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">T</div> <div style="display: inline-block; border: 1px solid black; padding: 2px 5px; margin: 0 5px;">F</div> | If a function $f(x, y)$ satisfies $ \nabla f(x, y) = 1$ everywhere in the plane, then f is constant. |

Problem 2) (10 points)



a) The picture above shows a contour map of a function $f(x, y)$ of two variables. This function has 12 critical points and all of them are marked. Each of them is either a local max, a local min or a saddle point. The picture shows also some gradient vectors. Count the number of critical points in the following table. No justifications are necessary.

The function $f(x, y)$ has		local maxima
The function $f(x, y)$ has		local minima
The function $f(x, y)$ has		saddle points

b) (4 points) Match the following partial differential equations with the names. No justifications are needed.

Enter A,B,C,D here	PDE
	$u_{xx} + u_{yy} = 0$
	$u_{xx} - u_{yy} = 0$

Enter A,B,C,D here	PDE
	$u_x - u_{yy} = 0$
	$u_x - u_y = 0$

A) Wave equation	B) Heat equation	C) Transport equation	D) Laplace equation
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Problem 3) (10 points)

Find the cos of the angle between the sphere

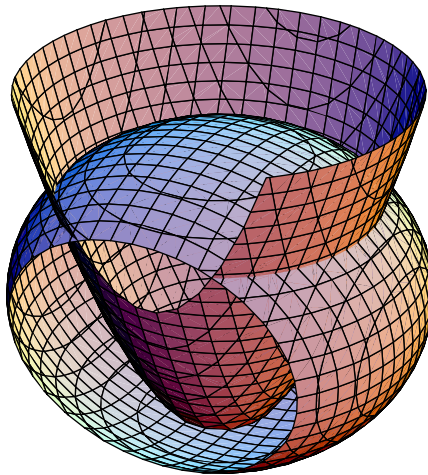
$$x^2 + y^2 + z^2 - 9 = 0$$

and the paraboloid

$$z - x^2 - y^2 + 3 = 0$$

at the point $(2, -1, 2)$.

Note: The angle between two general surfaces at a point P is defined as the angle between the tangent planes at the point P .



Problem 4) (10 points)

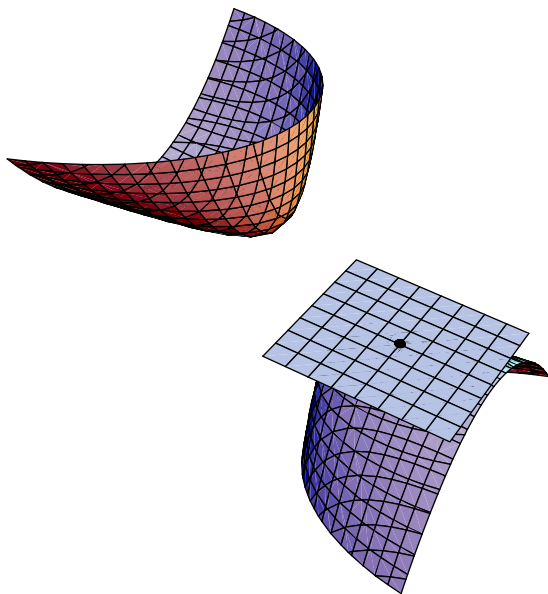
a) You know that

$$-2x + 5y + 10z = 2$$

is the equation of the tangent plane to the graph of $f(x, y)$ at the point $(-1, 2, -1)$.

Find the gradient $\nabla f(-1, 2)$ at the point $(-1, 2)$ and Estimate $f(-0.998, 2.0001)$ using linear approximation.

b) Let $f(x, y, z) = x^2 + 2y^2 + 3xz + 2$. Find the equation of the tangent plane to the surface $f(x, y, z) = 0$ at the point $(2, 0, -1)$ and estimate $f(2.001, 0.01, -1.0001)$.



Problem 5) (10 points)

- a) (4 points) Find all the critical points of the function $f(x, y) = xy$ in the interior of the elliptic domain

$$x^2 + \frac{1}{4}y^2 < 1.$$

and decide for each point whether it is a maximum, a minimum or a saddle point.

- b) (4 points) Find the extrema of f on the boundary

$$x^2 + \frac{1}{4}y^2 = 1.$$

of the same domain.

- c) (2 points) What is the global maximum and minimum of f on $x^2 + \frac{1}{4}y^2 \leq 1$.

Problem 6) (10 points)

- a) Assume $f(x, y) = e^{2x-y-2} + y + \sin(x-1)$ and $x(t) = \cos(5t)$, $y(t) = \sin(5t)$. What is

$$\frac{d}{dt}f(x(t), y(t))$$

at time $t = 0$.

b) The relation

$$xyz + z^3 + xy + yz^2 = 4$$

defines z as a function of x and y near $(x, y, z) = (1, 1, 1)$. Find the gradient

$$\left\langle \frac{\partial z}{\partial x}(1, 1), \frac{\partial z}{\partial y}(1, 1) \right\rangle$$

of $z(x, y)$ at the point $(1, 1)$.

Problem 7) (10 points)

The temperature in a room is given by $T(x, y, z) = x^2 + 2y^2 - 3z + 1$.

a) Barry B. Benson is hovering at the point $(1, 0, 0)$ and feels cold. Which direction should he go to heat up most quickly? Make sure that your answer is a unit vector.

b) At some later time, Barry arrives at the point $(3, 2, 1)$ and decides that this is a nice temperature. Find a direction (a unit vector) in which he can go, to stay at the same temperature and the same altitude.



Problem 8) (10 points)

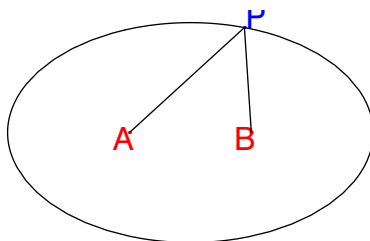
Let $g(x, y)$ denote the distance of a point $P = (x, y)$ to a point A and $h(x, y)$ the distance from P to a point B . The set of points (x, y) for which $f(x, y) = g(x, y) + h(x, y)$ is constant, forms an ellipse. In other words, the level curves of f are ellipses.

a) (4 points) Why is $\nabla g + \nabla h$ perpendicular to the ellipse?

b) (3 points) Show that if $\vec{r}(t)$ parametrizes the ellipse, then $(\nabla g + \nabla h) \cdot \vec{r}' = 0$ or $\nabla g \cdot \vec{r}' = -\nabla h \cdot \vec{r}'$.

c) (3 points) Conclude from this that the lines AP and BP make equal angles with the tangent to the ellipse at P . (Hint: check that $|\nabla f| = |\nabla g| = 1$).

You have now shown that light rays originating at focus A will be reflected from the ellipse to focus at the point B.

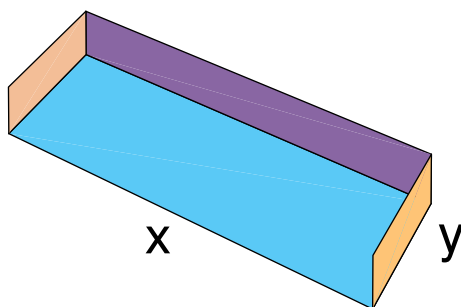


Problem 9) (10 points)

Minimize the material cost of an office tray

$$f(x, y) = xy + 2x + 2y$$

of length x , width y and height 1 under the constraint that the volume $g(x, y) = xy$ is constant and equal to 4.



Problem 10) (10 points)

A beach wind protection is manufactured as follows. There is a rectangular floor $ACBD$ of length a and width b . A pole of height c is located at the corner C and perpendicular to the ground surface. The top point P of the pole forms with the corners A and C one

triangle and with the corners B and C an other triangle. The total material has a fixed area of $g(a, b, c) = ab + ac/2 + bc/2 = 12$ square meters. For which dimensions a, b, c is the volume $f(a, b, c) = abc/6$ of the tetrahedral protected by this configuration maximal?

