11/7/2013 SECOND HOURLY PRACTICE VII **Math 21a, Fall 2013**

Name:

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TTH 10 Francesco Cavazzani
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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T F The directional derivative $D_u f$ is a vector normal to a level surface of f.

2) T F At a critical point of a function f, the gradient vector has length 1.

3) T F At a critical point (x, y) of a function f, the tangent plane to the graph of f does not exist.

4) T F For any point (x, y) which is not a critical point, there is a unit vector $\vec{u}$ for which $D_{\vec{u}} f(x, y)$ is nonzero.

5) T F If $f_{xx}(0,0) = 0, D = f_{xx}f_{yy} - f_{xy}^2 \neq 0$, and $\nabla f(0,0) = \langle 0, 0 \rangle$, then $(0,0)$ is a saddle point.

6) T F A continuous function defined on the closed unit disc $x^2 + y^2 \leq 1$ has an absolute maximum inside the disc or on the boundary.

7) T F The function $f(x, y) = x^2 - y^2$ has a neither a local maximum nor a local minimum at $(0, 0)$.

8) T F If (x, y) is a maximum of $f(x, y)$ under the constraint $g(x, y) = 5$ then it is also a maximum of $f(x, y) + g(x, y)$ under the constraint $g(x, y) = 5$.

9) T F The functions $f(x, y)$ and $g(x, y) = (f(x, y))^6$ always have the same critical points.

10) T F For $f(x, y, z) = x^2 + y^2 + 2z^2$, the vector $\nabla f(1, 1, 1)$ is perpendicular to the surface $f(x, y, z) = 4$ at the point $(1, 1, 1)$.

11) T F $f(x, y) = \sqrt{16 - x^2 - y^2}$ has both an absolute maximum and an absolute minimum on its domain of definition.

12) T F If (x_0, y_0) is a critical point of $f(x, y)$ and $f_{xy}(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point of f.

13) T F If $(1, 1, 1)$ is a maximum of f under the constraints $g(x, y, z) = c, h(x, y, z) = d$, and the Lagrange multipliers satisfy $\lambda = 0, \mu = 0$, then $(1, 1, 1)$ is a critical point of f.

14) T F Suppose f has a maximum value at a point P relative to the constraint $g = 0$. If the Lagrange multiplier $\lambda = 0$, then P is also a critical point for f without the constraint.

15) T F At a saddle point, all directional derivatives are zero.

16) T F The minimum of $f(x, y)$ under the constraint $g(x, y) = 0$ is always the same as the maximum of $g(x, y)$ under the constraint $f(x, y) = 0$.

17) T F At a local maximum (x_0, y_0) of $f(x, y)$, one has $f_{yy}(x_0, y_0) \leq 0$.

18) T F It is possible that $f(x, y)$ attains a maximum under the constraint $g(x, y) = 0$ at a point, where $\nabla f \neq \lambda \nabla g$.

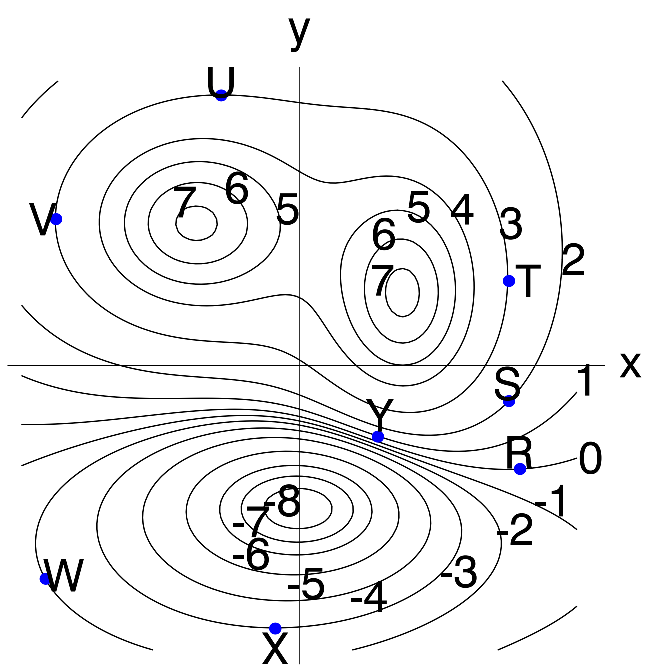
19) T F Any Lagrange problem which asks for an extremum of $f(x, y)$ under a constraint $g(x, y) = 0$ has either a maximum or a minimum.

20) T F The function $u(x, y) = \sin(x + y)$ satisfies the PDE $u_{xx} + u_{yy} - 2u_{xy} = 0$.

Problem 2) (10 points) No justifications needed.

a) (4 points) Fill in the boxes. You do not need to give additional explanations.

Chain rule:	$\frac{d}{dt}f(\vec{r}(t)) = \boxed{} \cdot \vec{r}\,'(t)$
Directional derivative D_v	$D_{\langle 2,3\rangle/\sqrt{13}} f(1,1) = \nabla f(1,1) \cdot \boxed{}$
Linearization of $f(x,y)$ at $(1,1)$	$L(x,y) = \boxed{} + \nabla f(1,1) \cdot (x-1, y-1)$
Equation of tangent line at $(1,1)$	$\nabla f(1,1) \cdot \langle x-1, y-1 \rangle = \boxed{}$
Critical point $(1,1)$ of f	$\nabla f(1,1) = \boxed{}$
Lagrange equations	$\nabla f(x,y) = \boxed{} \nabla g(x,y),\ g(x,y) = c.$
Type I integral	$\int_a^b \int_{c(x)}^{d(x)} f(x,y) \boxed{}$.
Type II integral	$\int_c^d \int_{a(y)}^{b(y)} f(x,y) \boxed{}$.
Integration in polar coordinates	$\int_a^b \int_{f(\theta)}^{g(\theta)} \boxed{} f(r\cos(\theta), r\sin(\theta))\, dr d\theta.$
Area	$\int\int_R \boxed{}\ dxdy$



b) (2 points) Circle the point at which the magnitude of the gradient vector ∇f is greatest. Mark exactly one point. Justify your answer.

R	S	T	U	V	W	X	Y

c) (2 points) Circle the points at which the partial derivative f_x is strictly positive. Mark any number of points on this question. Justify your answers.

R	S	T	U	V	W	X	Y

d) (2 points) We know that the directional derivative in the direction $(1,1)/\sqrt{2}$ is zero at one of the following points. Which one? Mark exactly one point on this question.

R	S	T	U	V	W	X	Y

Problem 3) (10 points)

a) Locate and classify all the critical points of

$$f(x,y) = 3y - y^3 - 3x^2y \ .$$

b) Where on the parameterized surface

$$\vec{r}(x,y) = \langle u,v,w \rangle = \langle xy^3, x^2/2, 3y^2/2 \rangle$$

is the function $g(u,v,w) = u - v - w$ extremal? To investigate this, find all the critical points of the function $f(x,y) = xy^3 - \frac{x^2}{2} - \frac{3y^2}{2}$. For each critical point, specify whether it is a local maximum, a local minimum or a saddle point and show how you know.

Problem 4) (10 points)

Evaluate the double integral

$$\int_0^4 \int_0^{y^2} \frac{x^4}{4 - \sqrt{x}} \, dx \, dy \ .$$

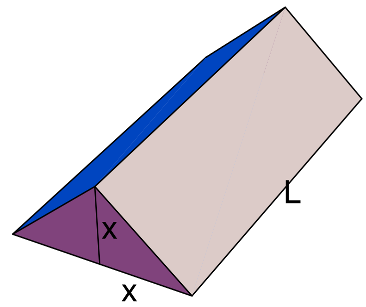
Problem 5) (10 points)

a) (6 points) Find all critical points of $f(x,y) = 3xe^y - e^{3y} - x^3$ and classify them.

b) (4 points) Does the function have an absolute maximum or absolute minimum? Make sure to justify also this answer.

Problem 6) (10 points)

We minimize the surface of a roof of height x and width $2x$ and length $L = \sqrt{2}y$ if the volume $V(x,y) = x^2\sqrt{2}y$ of the roof is fixed and equal to $\sqrt{2}$. In other words, you have to minimize $f(x,y) = 2x^2 + 4xy$ under the constraint $g(x,y) = x^2y = 1$. Solve the problem with the Lagrange method.



Problem 7) (10 points)

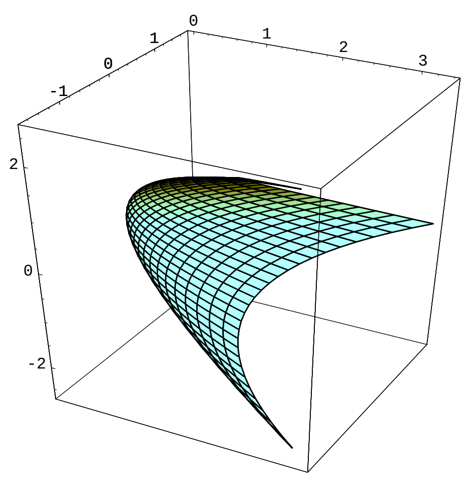
Find all the critical points of $f(x,y) = \frac{x^5}{5} - \frac{x^2}{2} + \frac{y^3}{3} - y$ and indicate whether they are local maxima, local minima or saddle points.

Problem 8) (10 points)

The temperature distribution in a room is $T(x,y,z) = x + y + z$. On which point of the parametrized surface

$$\vec{r}(s,t) = \langle x,y,z \rangle = \langle s^2 + t^2, st, 2s - t \rangle$$

is the temperature extremal? Is it a maximum or a minimum?



Problem 9) (10 points)

A region R in the xy-plane is given in polar coordinates by $r(\theta) \leq \theta^2$ for $\theta \in [0, \pi]$. You see the region in the picture to the right. Evaluate the double integral

$$\iint_R \frac{\cos(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}(\pi - (x^2 + y^2)^{1/4})} \, dx \, dy \ .$$

Problem 10) (10 points)

Suppose $2x + 3y + 2z = 9$ is the tangent plane to the graph of $z = f(x, y)$ at the point $(1, 1, 2)$.

a) Find the linear approximation of $f(1.01, 0.98)$.
b) What is the gradient ∇f at $(1, 1)$?
c) What is the equation $ax + by = d$ of the tangent line at $(1, 1)$?