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TTH 10 Oliver Knill
TTH 10 Francesco Cavazzani
TTH 11:30 Kate Penner
TTH 11:30 Francesco Cavazzani

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

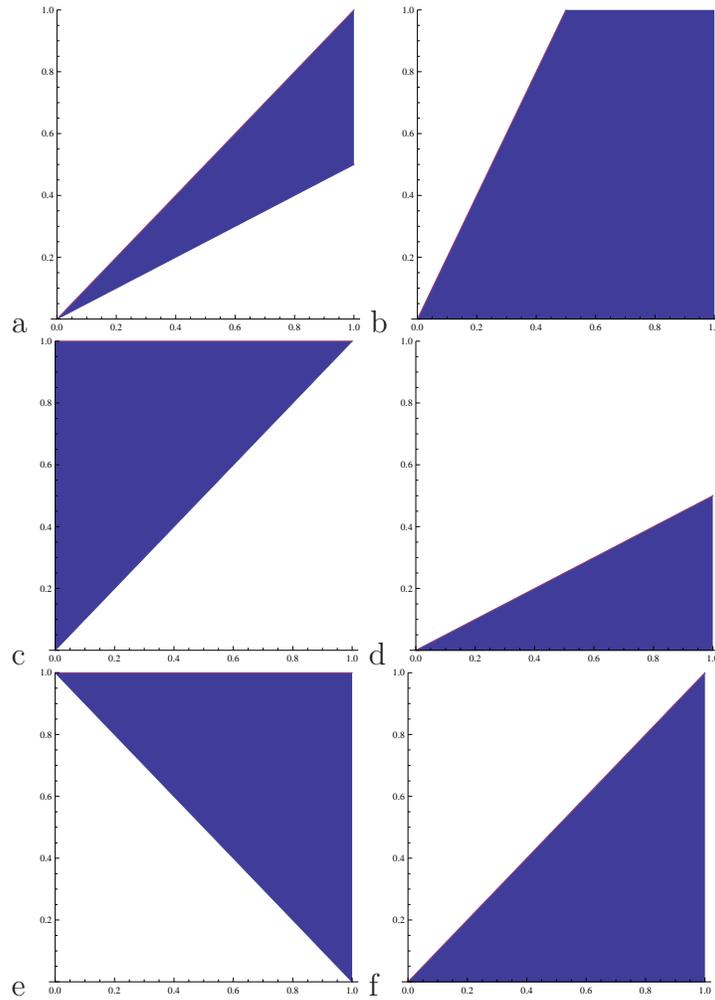
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

- 1) T F The directional derivative $D_{\vec{v}}f$ is a vector perpendicular to \vec{v} .
- 2) T F Using linearization of $f(x, y) = xy$ we can estimate $f(0.9, 1.2) \sim 1 - 0.1 + 0.2 = 1.1$.
- 3) T F Given a curve $\vec{r}(t)$ on a surface $g(x, y, z) = 1$, then $\frac{d}{dt}g(\vec{r}(t)) = 0$.
- 4) T F Given a function $f(x, y)$ such that $\nabla f(0, 0) = \langle 2, -1 \rangle$. Then $D_{\langle 0, -1 \rangle}f(0, 0) = 0$.
- 5) T F $\vec{r}(u, v) = \langle u \cos(v), u \sin(v), v \rangle$ is a surface of revolution.
- 6) T F If $(1, 1)$ is a critical point for the function $f(x, y)$ then $(1, 1)$ is also a critical point for the function $g(x, y) = f(x^2, y^2)$.
- 7) T F If $f(x, y)$ has a local maximum at $(0, 0)$ then it is possible that $f_{xx}(0, 0) > 0$ and $f_{yy}(0, 0) < 0$.
- 8) T F The integral $\int_0^x \int_0^y 1 \, dx dy$ computes the area of a region in the plane.
- 9) T F The function $f(x, y) = x^2 + y^4$ has a local minimum at $(0, 0)$.
- 10) T F The integral $\int_0^1 \int_0^1 x^2 + y^2 \, dx dy$ is the volume of the solid bounded by the 5 planes $x = 0, x = 1, y = 0, y = 1, z = 0$ and the paraboloid $z = x^2 + y^2$.
- 11) T F There exists a region in the plane, which is neither a type I integral, nor a type II integral.
- 12) T F Fubini's theorem assures that $\int_0^1 \int_0^x f(x, y) \, dy dx = \int_0^1 \int_0^y f(x, y) \, dx dy$.
- 13) T F The function $f(x, y) = \sin(x) \cos(y)$ satisfies the partial differential equation $f_{xx} + f_{yy} = 0$.
- 14) T F Let $L(x, y)$ be the linearization of $f(x, y) = \sin(x(y + 1))$ at $(0, 0)$. Then, the level curves of $L(x, y)$ consist of lines.
- 15) T F For any smooth function $f(x, y)$, the inequality $|\nabla f| \geq |f_x + f_y|$ is true.
- 16) T F Any differentiable function $f(x, y)$ which satisfies the partial differential equation $\|\nabla f\|^2 = 0$ is constant.
- 17) T F If $x + \sin(xy) = 1$, $dy/dx = \frac{-(1+y \cos(yx))}{(x \cos(xy))}$.
- 18) T F The directional derivative $D_v f(1, 1)$ is zero if v is a unit vector tangent to the level curve of f which goes through $(1, 1)$.
- 19) T F If (a, b) is a maximum of $f(x, y)$ under the constraint $g(x, y) = 0$, then the Lagrange multiplier λ there has the same sign as the discriminant $D = f_{xx}f_{yy} - f_{xy}^2$ at (a, b) .
- 20) T F If $D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle}f(1, 2) = 0$ and $D_{\langle -1/\sqrt{2}, 1/\sqrt{2} \rangle}f(1, 2) = 0$, then $(1, 2)$ is a critical point.

Problem 2) (10 points)

Match the regions with the corresponding double integrals



Enter a,b,c,d,e or f	Integral of Function $f(x, y)$
	$\int_0^1 \int_{x/2}^x f(x, y) dy dx$
	$\int_0^1 \int_0^1 f(x, y) dx dy$
	$\int_0^1 \int_0^{x/2} f(x, y) dy dx$
	$\int_0^1 \int_{y/2}^1 f(x, y) dx dy$
	$\int_0^1 \int_0^x f(x, y) dy dx$
	$\int_0^1 \int_{1-x}^1 f(x, y) dy dx$

Problem 3) (10 points)

Let $g(x, y, z) = x^2 + 2y^2 - z - 3$.

- a) (5 points) Find the equation of the tangent plane to the level surface $g(x, y, z) = 0$ at the point $(x_0, y_0, z_0) = (2, 0, 1)$.
- b) (5 points) The surface in a) is the graph $z = f(x, y)$ of a function of two variables. Find the tangent line to the level curve $f(x, y) = 1$ at the point $(x_0, y_0) = (2, 0)$.

Problem 4) (10 points)

- a) (5 points) Use the technique of linear approximation to estimate $f(\pi/2 + 0.1, 2.9)$ for

$$f(x, y) = (10 \sin(x) - 5y^2 + 8)^{1/3}.$$

- b) (5 points) Find the unit vector at $(\pi/2, 3)$, in the direction where the function increases fastest.

Problem 5) (10 points)

The pressure in the space at the position (x, y, z) is $p(x, y, z) = x^2 + y^2 - z^3$ and the trajectory of an observer is the curve $\vec{r}(t) = \langle t, t, 1/t \rangle$.

- a) (2 points) State the chain rule which applies in this situation.
- b) (4 points) Using the chain rule in a) compute the rate of change of the pressure the observer measures at time $t = 2$.
- c) (4 points) At which time t does the observer go in the direction, in which the pressure decreases most?

Problem 6) (10 points)

The coffee chain **Astrbucks**¹ has branches at $(0, 0)$, $(0, 3)$ and $(3, 3)$ (JFK street, Church street, and Broadway) near Harvard square. A caffeine addicted [politically correct: loving] mathematician wants to rent an apartment at a location, where the sum of the squared distances $f(x, y)$ to all those shops is a local minimum. The function is

$$f(x, y) = (x-0)^2 + (y-0)^2 + (x-0)^2 + (y-3)^2 + (x-3)^2 + (y-3)^2 = 27 - 6x + 3x^2 - 12y + 3y^2.$$

- a) (5 points) Where does the mathematician have to live to locally minimize $f(x, y)$?
- b) (3 points) For every local minimum answer: Is this local minimum a **global** minimum?
- c) (2 points) Is there a global maximum to this problem? If yes, give it. If no, why not?

¹This problem was sponsored by *Astrbucks*©.

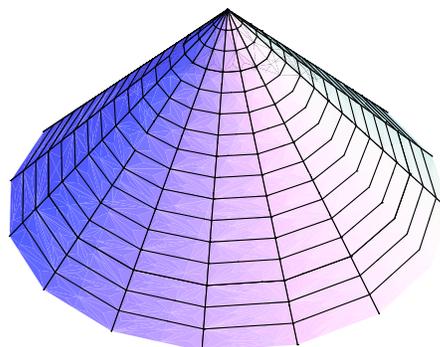


Problem 7) (10 points)

Find all the critical points of $f(x, y) = 3xy + x^2y + xy^2$ and classify them as saddle points, local maxima or local minima.

Problem 8) (10 points)

A solid cone of height h and with base radius r has the volume $f(h, r) = \frac{h\pi r^2}{3}$ and the surface area $g(h, r) = \pi r\sqrt{r^2 + h^2} + \pi r^2$. Among all cones with fixed surface area $g(h, r) = \pi$ use the Lagrange method to find the cone with maximal volume.



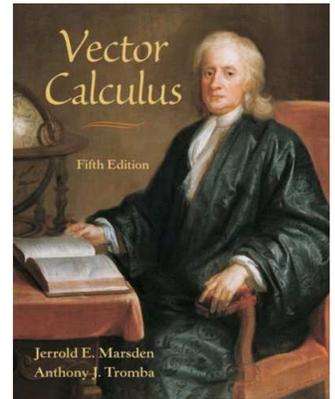
Problem 9) (10 points)

Marsden and Tromba pose in their textbook the following riddle:
Suppose $w = f(x, y)$ and $y = x^2$. By the chain rule

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial x} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial x} + 2x \frac{\partial w}{\partial y}$$

so that $0 = 2x \frac{\partial w}{\partial y}$ and so $\frac{\partial w}{\partial y} = 0$.

- Find an explicit example of a function $f(x, y)$, where you see the argument is false.
- What is flawed in the above application of the chain rule?



Problem 10) (10 points)

Evaluate the double integral

$$\int \int_R \sqrt{x^2 + y^2} \, dx dy$$

where R is the region bounded by the positive x -axis, the spiral curve $\vec{r}(t) = \langle t \cos(t), t \sin(t) \rangle$, $0 \leq t \leq 2\pi$ and the circle with radius 2π .

