

Name:

MWF 9 Oliver Knill
MWF 9 Chao Li
MWF 10 Gijs Heuts
MWF 10 Adrian Zahariuc
MWF 10 Yihang Zhu
MWF 11 Peter Garfield
MWF 11 Matthew Woolf
MWF 12 Charmaine Sia
MWF 12 Steve Wang
MWF 14 Mike Hopkins
TTH 10 Oliver Knill
TTH 10 Francesco Cavazzani
TTH 11:30 Kate Penner
TTH 11:30 Francesco Cavazzani

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

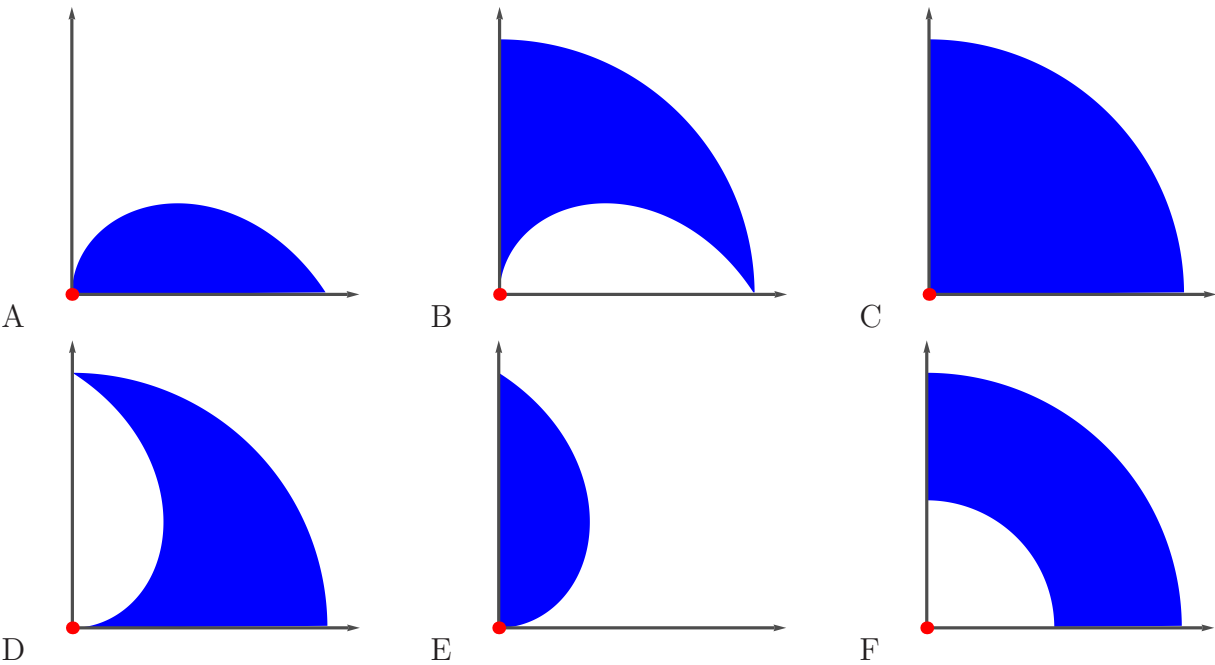
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

- 1) ☐ T ☐ F If $f(x, y) = 1$ is a curve, and near $(2, 3)$ one can write y as a function of x , then $y' = -f_y(2, 3)/f_x(2, 3)$.
- 2) ☐ T ☐ F If $\iint_R f(x, y) \, dA = 0$, then the function $f(x, y)$ is everywhere zero on $R = \{x^2 + y^2 \leq 1\}$.
- 3) ☐ T ☐ F The directional derivative in the direction of the gradient is $|\nabla f|$.
- 4) ☐ T ☐ F The linearization of $f(x, y) = x^3 + y^3$ at $(1, 1)$ is the quadratic function $L(x, y) = 3x^2 + 3y^2$.
- 5) ☐ T ☐ F The function $f(x, y) = x^2 + y^2$ satisfies the partial differential equation $D = f_{xx}f_{yy} - f_{xy}^2 = 1$.
- 6) ☐ T ☐ F The function x^2y^2 has no local minimum at $(0, 0)$ because the discriminant function D is zero there.
- 7) ☐ T ☐ F The double integral $\int_0^{\pi/4} \int_0^2 r^3 \, dr d\theta$ is the volume of the part of a solid cylinder $x^2 + y^2 \leq 4$ which is below the paraboloid $z = x^2 + y^2$ and above the xy plane.
- 8) ☐ T ☐ F The gradient of $f(x, y, z)$ at (x_0, y_0, z_0) is perpendicular to the level surface of f through (x_0, y_0, z_0) .
- 9) ☐ T ☐ F If $f(x, y, z) = 3x - 4z$, then the minimal possible directional derivative $D_{\vec{u}}f$ at any point in space is -5 .
- 10) ☐ T ☐ F If (x, y) is not a critical point, then the directional derivative $D_{\vec{v}}f$ can take both positive and negative values for different choices of $\vec{v}$.
- 11) ☐ T ☐ F Using linearization of $f(x, y) = x/y$ we can estimate $1.01/1.001 = f(1.01, 1.001) \sim 1 + 0.01 - 0.001 = 1.009$.
- 12) ☐ T ☐ F If $(0, 0)$ is a critical point of $f(x, y)$ with nonzero discriminant $D = f_{xx}f_{yy} - f_{xy}^2$, we know that it is either a saddle, a global maximum or a global minimum.
- 13) ☐ T ☐ F For a rectangular region R , Fubini tells that $\int_0^2 \int_0^3 f(x, y) \, dx dy = \int_0^2 \int_0^3 f(x, y) \, dy dx$ for any continuous function $f(x, y)$.
- 14) ☐ T ☐ F If a function $f(x, y)$ has only one critical point $(0, 0)$ in $G = \{x^2 + y^2 \leq 1\}$ which is a local maximum and $f(0, 0) = 1$, then $\iint_G f(x, y) \, dx dy > 0$.
- 15) ☐ T ☐ F If $\vec{r}(t)$ is a curve in space for which the speed is 1 at all times and $f(x, y, z)$ is a function of three variables, then $d/dt f(\vec{r}(t)) = D_{\vec{r}'(t)}(f)$.
- 16) ☐ T ☐ F $\int_0^1 \int_0^1 f_{xy}(x, y) \, dy dx = f(1, 1) - f(1, 0) - f(0, 1) + f(0, 0)$.
- 17) ☐ T ☐ F If $f_{yy}(x, y) > 0$ everywhere, then f can not have any local maximum.
- 18) ☐ T ☐ F The double integral $\int_0^1 \int_0^1 x^2 - y^2 \, dx dy$ is the volume of the solid below the graph of $f(x, y) = x^2 - y^2$ and above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ in the xy -plane.
- 19) ☐ T ☐ F For any unit vector $\vec{v}$ and any differentiable function f , one has $D_{\vec{v}}(f) + D_{-\vec{v}}(f) = 0$.
- 20) ☐ T ☐ F The surfaces $x + y + z = 0$ and $x^2 + y^2 + z^2 + x + y + z = 0$ have the same tangent plane at $(0, 0, 0)$.

Problem 2) (10 points)

a) (6 points) Match the regions with the corresponding polar double integrals



Enter A-F	Integral of $f(r, \theta)$	Enter A-F	Integral of $f(r, \theta)$
	$\int_0^{\pi/2} \int_0^{\pi/2} f(r, \theta) r \, dr d\theta$		$\int_0^{\pi/2} \int_{\theta}^{\pi/2} f(r, \theta) r \, dr d\theta$
	$\int_0^{\pi/2} \int_0^{\theta} f(r, \theta) r \, dr d\theta$		$\int_0^{\pi/2} \int_{\pi/2-\theta}^{\pi/2} f(r, \theta) r \, dr d\theta$
	$\int_0^{\pi/2} \int_0^{\pi/2-\theta} f(r, \theta) r \, dr d\theta$		$\int_0^{\pi/2} \int_{\pi/4}^{\pi/2} f(r, \theta) r \, dr d\theta$

b) (4 points) Match the partial differential equations (PDE's) for the functions $u(t, s)$ with their names. No justifications are needed.

Enter A,B,C,D here	PDE	Enter A,B,C,D here	PDE
	$u_t + uu_s - u_{ss} = 0$		$u_{tt} - u_{ss} = 0$
	$u_{tt} + u_{ss} = 0$		$u_t - u_{ss} = 0$

A) Wave equation

B) Heat equation

C) Burgers equation

D) Laplace equation

Problem 3) (10 points)

a) (7 points) Find and classify all the critical points of the function

$$f(x, y) = 5 + 3x^2 + 3y^2 + y^3 + x^3 .$$

b) (3 points) Is there a global maximum or a global minimum for $f(x, y)$?

Problem 4) (10 points)

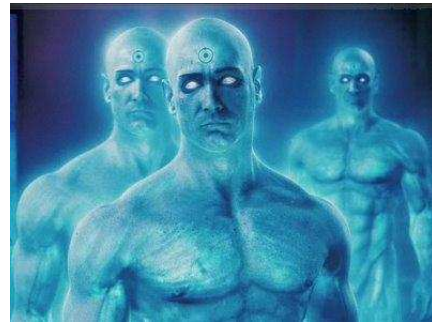
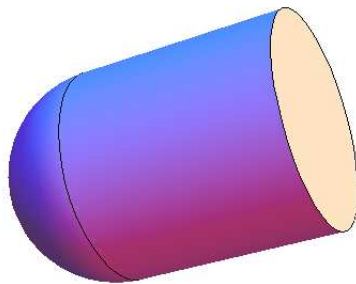
A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi r h + \pi r^2$. Doctor Manhattan designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes

$$f(h, r) = 3r^2 + 2rh$$

under the constraint

$$g(h, r) = 2r^3 + 3r^2 h = 1 .$$

Use the Lagrange method to find a local minimum of f under the constraint $g = 1$.

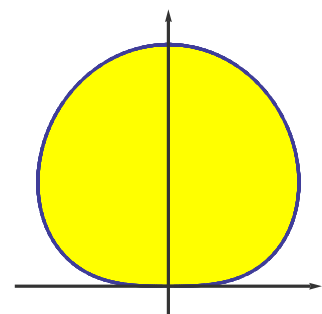


Problem 5) (10 points)

A region R in the plane shown to the right is called the “**blob of nothingness**”. It does not have any purpose nor meaning. It just sits there. The region is given in polar coordinates as $0 \leq r \leq \theta(\pi - \theta)$ for $0 \leq \theta \leq \pi$. Find the area

$$\int \int_R 1 \, dx dy$$

of this nihilistic object.



Problem 6) (10 points)

a) (4 points) If

$$f(x, y) = y \cos(x - y) ,$$

find equation of plane tangent to $z = f(x, y)$ at the point $(2, 2, 2)$.

b) (3 points) Find the equation of the tangent line to $f(x, y) = 2$ at $(2, 2)$.

c) (3 points) Estimate $f(2.1, 1.9)$ using linear approximation.

Problem 7) (10 points)

A **Harvard robot bee** flies along the curve

$$\vec{r}(t) = \langle t - t^3, 3t^2 - 3t \rangle$$

and measures the temperature $f(x, y)$. It flies over the target point $(0, 0)$ at time $t = 0$ and time $t = 1$. At each time, its sensor measures the temperature change $g'(t)$ where $g(t) = f(\vec{r}(t))$.

a) (5 points) Assume you knew that the gradient of f at $(0, 0)$ is $\langle a, b \rangle$. What are the values of $g'(t) = d/dt f(\vec{r}(t))$ at $t = 0$ and $t = 1$ in terms of a and b ?

b) (5 points) The bee measures $g'(0) = 3$ and $g'(1) = 3$. What is the gradient $\nabla f(0, 0) = \langle a, b \rangle$ of f at $(0, 0)$?

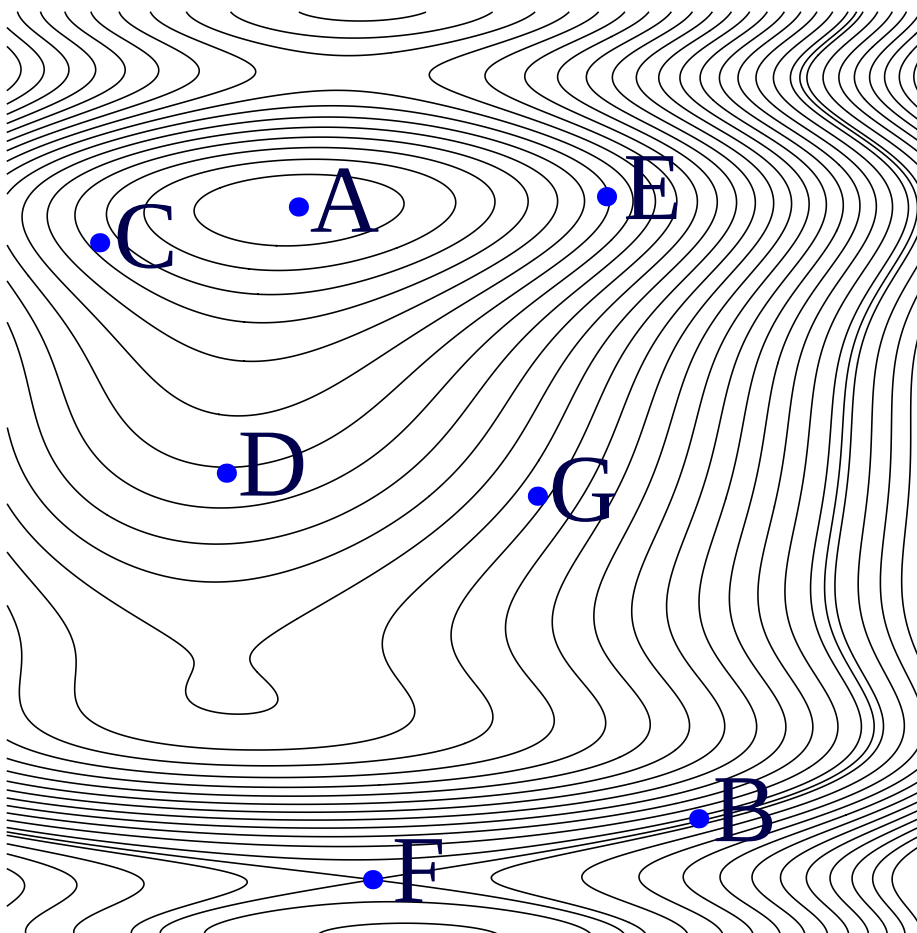


Image source: Harvard Press release on
robobees.seas.harvard.edu

Problem 8) (10 points)

A function $f(x, y)$ of two variables has level curves as shown in the picture. The function values at neighboring level curves differ by 1. [No justifications are needed in this problem. Naturally, since there are less points than boxes, some of the points A-G will appear more than once, but each box will only be filled with one letter.]

Enter A-G	is a point, where ...
	$f_x(x, y) = 0$ and $f_y(x, y) \neq 0$.
	$f_y(x, y) = 0$ and $f_x(x, y) \neq 0$.
	$f(x, y)$ has either a max or a min.
	$f(x, y)$ has a saddle point.
	$f(x, y)$ has no max nor min but is extremal under a constraint $y = c$ for some c .
	$f(x, y)$ has no max nor min but is extremal under a constraint $x = c$ for some c .
	the length of the gradient vector of f is largest among all points A-G.
	$D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f(x, y) = 0$ and $D_{\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle} f(x, y) \neq 0$.
	$D_{\langle 1/\sqrt{2}, -1/\sqrt{2} \rangle} f(x, y) = 0$ and $D_{\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle} f(x, y) \neq 0$.
	the tangent line to the curve is $x + y = d$ for some constant d .



Problem 9) (10 points)

Evaluate the following double integral

$$\int_0^1 \int_0^{(1-x)^2} \frac{x^3}{(1-\sqrt{y})^4} dy dx .$$

Problem 10) (10 points)

A mass point with position (x, y) is attached by springs to the points $A_1 = (0, 0)$, $A_2 = (2, 0)$, $A_3 = (0, 2)$, $A_4 = (2, 3)$, $A_5 = (3, 1)$. It has the potential energy

$$f(x, y) = 31 - 14x + 5x^2 - 12y + 5y^2$$

which is the sum of the squares of the distances from (x, y) to the 5 points. Find all extrema of f using the second derivative test. The minimum of f is the position, where the mass point has the lowest energy.

