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TTH 11:30 Kate Penner
TTH 11:30 Francesco Cavazzani

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

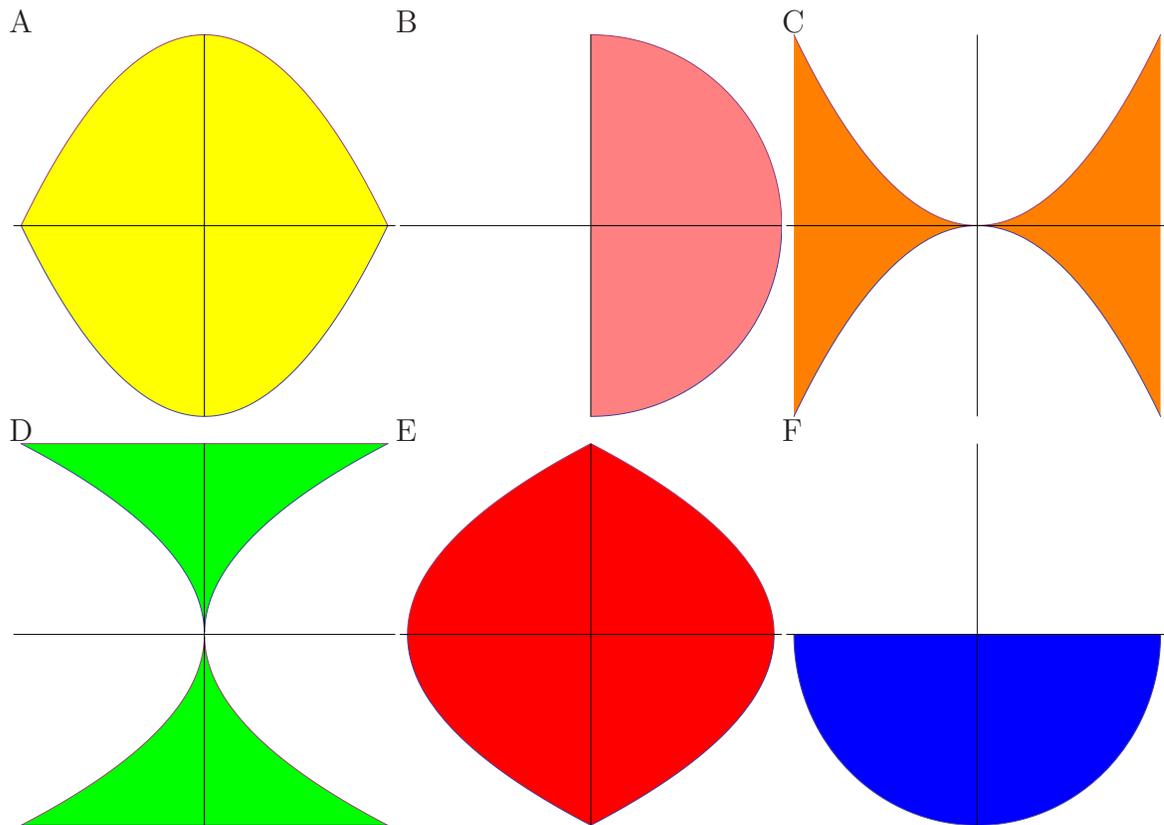
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

- 1)  T  F There is a function  $f(x, y)$  for which the linearization at  $(0, 0)$  is  $L(x, y) = x^2 + y^2$ .
- 2)  T  F For any two functions  $f, g$  and unit vector  $\vec{u}$  we have  $D_{\vec{u}}(f + g) = D_{\vec{u}}f + D_{\vec{u}}g$ .
- 3)  T  F  $\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2 + y^2) dydx = \int_0^2 \int_0^{\pi/2} r^2 d\theta dr$ .
- 4)  T  F If we solve  $\sin(y) - xy^2 = 0$  for  $y$ , then  $y' = -y^2/(\cos(y) - 2xy)$ .
- 5)  T  F If  $f(x, 0) = 0$  for all  $x$  and  $f(0, y) = 0$  for all  $y$ , then  $g(x, y) = \int_0^x \int_0^y f(s, t) dt ds$  solves  $g_{xy}(x, y) = f(x, y)$ .
- 6)  T  F If  $|\nabla f| = 1$  at  $(0, 0)$ , then there exists a direction in which the slope of the graph of  $f$  at  $(0, 0)$  is 1.
- 7)  T  F The function  $f(x, y) = x^2 + y^2$  satisfies the partial differential equation  $f_{xx}f_{yy} - f_{xy}^2 = 4$ .
- 8)  T  F The height of Mount Wachusett is  $f(x, y) = 4 - 2x^2 - y^2$ . On the trail  $x^2 + y^2 = 1$ , the point  $(1, 0)$  is a maximum.
- 9)  T  F Mount Wachusett has height  $f(x, y) = 4 - 2x^2 - y^2$ . Except at the maximum  $(0, 0)$ , the gradient vector is perpendicular to the graph of the function.
- 10)  T  F If  $f_x(a, b) > 0$  and  $f_y(a, b) > 0$  then for any unit vector  $\vec{u}$  we must have  $D_{\vec{u}}f(a, b) > 0$ .
- 11)  T  F If  $f(x, y)$  has two local minima, then  $f$  must have at least one local maximum.
- 12)  T  F If  $\vec{r}(t)$  is a curve on the surface  $g(x, y, z) = x^2 + y^2 - z^2 = 6$  then  $\nabla g(\vec{r}(t)) \cdot \vec{r}'(t) = 0$ .
- 13)  T  F If  $f$  and  $g$  have the same trace  $\{x = 5\}$  then  $f_x(5, y) = g_x(5, y)$  for all  $y$ .
- 14)  T  F If  $f$  and  $g$  have the same trace  $\{x = 5\}$  then  $f_y(5, y) = g_y(5, y)$  for all  $y$ .
- 15)  T  F The surface area of  $\vec{r}_1(u, v) = \langle u \cos(v), u \sin(v), u^2 \rangle$  and  $\vec{r}_2(u, v) = \langle \sqrt{u} \cos(v), \sqrt{u} \sin(v), u \rangle$  defined on  $\{0 \leq u, v \leq 1\}$  are the same.
- 16)  T  F If  $\vec{r}(t)$  is a curve on a graph  $z = f(x, y)$  of a function  $f(x, y)$ , then the velocity vector of  $\vec{r}$  is perpendicular to the vector  $\langle f_x, f_y, -1 \rangle$ .
- 17)  T  F A continuous function  $f(x, y)$  on the closed disc  $R = \{x^2 + y^2 \leq 51^2\}$  (of course,  $R$  is called “**area**  $51\pi$ ”) has a global maximum on  $R$ .
- 18)  T  F Any continuous function  $f(x, y)$  has a global minimum and maximum on the curve  $y = x^2$ .
- 19)  T  F Fubini’s theorem assures that  $\int_a^b \int_c^d f(x, y) dydx = \int_a^b \int_c^d f(x, y) dx dy$ .
- 20)  T  F  $\iint_R \sin(x + y) dx dy = 0$  for  $R = \{-\pi \leq x \leq \pi, -\pi \leq y \leq \pi\}$ .

Problem 2) (10 points)

a) (6 points) Match the integration regions with the integrals. Each integral matches exactly one region  $A - F$ .



Enter A-F	Integral
	$\int_{-1}^1 \int_{-x^2}^{x^2} f(x, y) dydx.$
	$\int_{-1}^1 \int_{-y^2}^{y^2} f(x, y) dx dy.$
	$\int_{-1}^1 \int_{y^2-1}^{1-y^2} f(x, y) dx dy.$
	$\int_{-1}^1 \int_0^{\sqrt{1-y^2}} f(x, y) dx dy.$
	$\int_{-1}^1 \int_{x^2-1}^{1-x^2} f(x, y) dy dx.$
	$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 f(x, y) dy dx.$

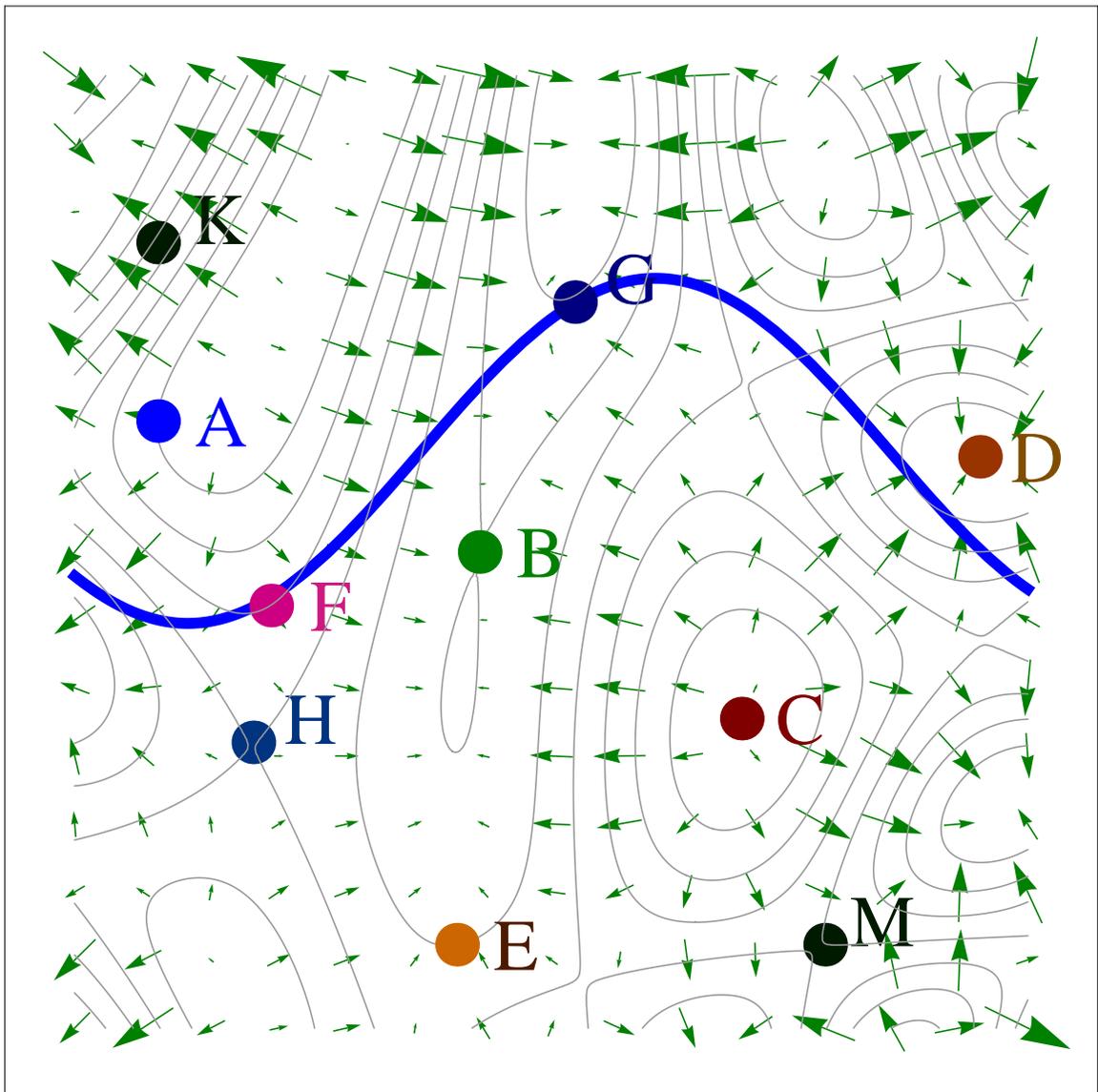
b) (4 points) Fill in one word names (like “Heat”, “Wave” etc) for the partial differential equations:

Enter one word	PDE
	$g_x = g_y$
	$g_{xx} = g_{yy}$
	$g_{xx} = -g_{yy}$
	$g_x = g_{yy}$

Problem 3) (10 points)

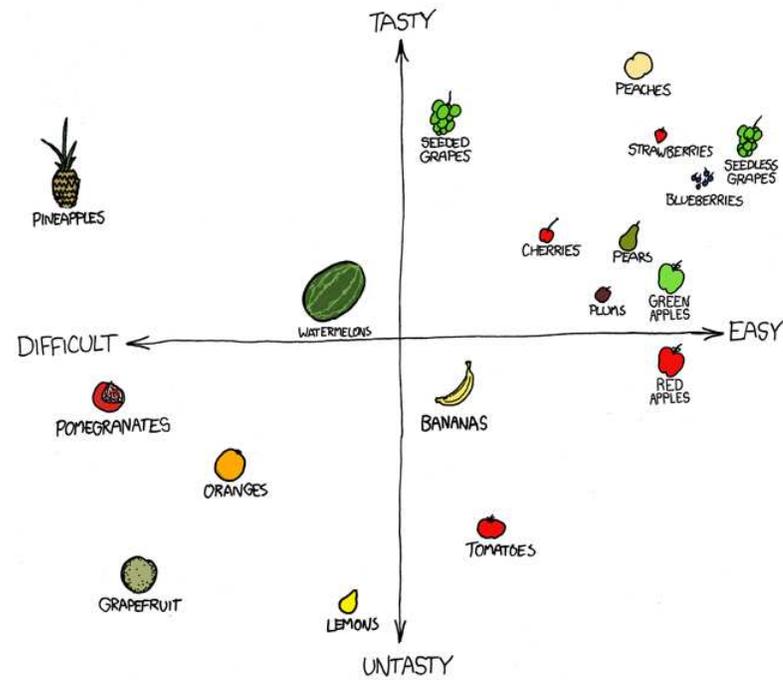
(10 points) A function  $f(x, y)$  of two variables has level curves as shown in the picture. We also see a constraint in the form of a curve  $g(x, y) = 0$  which has the shape of the graph of the cos function. The arrows show the gradient. In this problem, each of the 10 letters  $A, B, C, D, E, F, G, H, K, M$  appears exactly once.

Enter A-P	Description
	a local maximum of $f(x, y)$ .
	a local minimum of $f(x, y)$ .
	a saddle point of $f(x, y)$ where $f_{xx} < 0$ .
	a saddle point of $f(x, y)$ where $f_{xx} > 0$ .
	a saddle point of $f(x, y)$ where $f_{xx}$ is close to zero
	a point, where $f_x = 0$ and $f_y \neq 0$
	a point, where $f_y = 0$ and $f_x \neq 0$
	the point, where $ \nabla f $ is largest
	a local maximum of $f(x, y)$ under the constraint $g(x, y) = 0$ .
	a local minimum of $f(x, y)$ under the constraint $g(x, y) = 0$ .



Problem 4) (10 points)

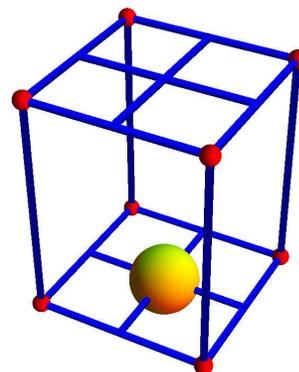
Find and classify all the extrema of the function  $f(x, y) = x^5 + y^3 - 5x - 3y$ . This function measures “eat temptation” in the  $x$ =Easy- $y$ =Tasty plane. Is there a global minimum or global maximum?



The “Easy-Tasty plane” was introduced in the XKCD cartoon titled “F&#% Grapefruits”.

Problem 5) (10 points)

After having watched the latest Disney movie “Tangled”, we want to build a hot air balloon with a cuboid mesh of dimension  $x, y, z$  which together with the top and bottom fortifications uses wires of total length  $g(x, y, z) = 6x + 6y + 4z = 32$ . Find the balloon with maximal volume  $f(x, y, z) = xyz$ .



Problem 6) (10 points)

a) (8 points) Find the tangent plane to the surface  $f(x, y, z) = x^2 - y^2 + z = 6$  at the point  $(2, 1, 3)$ .

b) (2 points) A curve  $\vec{r}(t)$  on that tangent plane of the function  $f(x, y, z)$  in a) has constant speed  $|\vec{r}'| = 1$  and passes through the point  $(2, 1, 3)$  at  $t = 0$ . What is  $\frac{d}{dt}f(\vec{r}(t))$  at  $t = 0$ ?

Problem 7) (10 points)

a) (5 points) Estimate  $\sqrt{\sin(0.0004) + 1.001^2}$  using linear approximation.

b) (5 points) We know  $f(0, 0) = 1$ ,  $D_{\langle \frac{3}{5}, \frac{4}{5} \rangle}f(0, 0) = 2$  and  $D_{\langle -\frac{4}{5}, \frac{3}{5} \rangle}f(0, 0) = -1$ . If  $L(x, y)$  is the linear approximation to  $f(x, y)$  at the point  $(0, 0)$ , find  $L(0.06, 0.08)$ .

Problem 8) (10 points)

a) (5 points) Find the following double integral

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \frac{\pi \sin(\pi y)}{y^2 - \sqrt{y}} dy dx .$$

b) (5 points) Evaluate the following double integral

$$\iint_R \frac{\sin(\pi \sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} dx dy$$

over the region

$$R = \{x^2 + y^2 \leq 1, x > 0\} .$$

Problem 9) (10 points)

a) (8 points) Find the surface area of the surface parametrized as

$$\vec{r}(u, v) = \langle u - v, u + v, (u^2 - v^2)/2 \rangle ,$$

where  $(u, v)$  is in the unit disc  $R = \{u^2 + v^2 \leq 1\}$ .

b) (2 points) Give a nonzero vector  $\vec{n}$  normal to the surface at  $\vec{r}(4, 2) = \langle 2, 6, 6 \rangle$ .

Problem 10) (10 points)

a) (6 points) Integrate

$$\int_0^{\pi/2} \int_x^{\pi/2} \frac{\cos(y)}{y} dy dx$$

b) (4 points) Find the moment of inertia

$$\iint_R (x^2 + y^2) dy dx ,$$

where  $R$  is the ring  $1 \leq x^2 + y^2 \leq 9$ .