10/2/2013 FIRST HOURLY PRACTICE VIII **Math 21a, Fall 2013**

Name:

MWF 9 Oliver Knill
MWF 9 Chao Li
MWF 10 Gijs Heuts
MWF 10 Adrian Zahariuc
MWF 10 Yihang Zhu
MWF 11 Peter Garfield
MWF 11 Matthew Woolf
MWF 12 Charmaine Sia
MWF 12 Steve Wang
MWF 14 Mike Hopkins
TTH 10 Oliver Knill
TTH 10 Francesco Cavazzani
TTH 11:30 Kate Penner
TTH 11:30 Francesco Cavazzani

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points)

Mark for each of the 20 questions the correct letter. No justifications are needed.

1) T **F** The surface described in spherical coordinates as $\phi = \pi/4$ is the xy plane.

Solution:
The xy-plane is $\phi = \pi/2$. Note that $\phi = \pi/4$ is the upper part of a cone.

2) **T** F The length of the unit tangent vector $\vec{T}$ for a curve $\vec{r}(t)$ is independent of t.

Solution:
It has length 1.

3) T **F** For all vectors $\vec{v}$ and $\vec{w}$ the vector $\vec{w} \times (\vec{w} \times \vec{v})$ is perpendicular to $\vec{v}$.

Solution:
Take an example like $\vec{v} = \vec{j}$ and $\vec{w} = i$.

4) **T** F There is a point (x, y, z) in space, for which the cylindrical coordinates (r, θ, z) and spherical coorindates (ρ, θ, ϕ) satisfy $(r, \theta, z) = (\rho, \theta, \phi - \pi/2)$.

Solution:
The first component mans $z = 0$. This implies $\phi = \pi/2$. Every point on the x, y plane satisfies this.

5) T **F** The two planes $x + y - z = 1$ and $-x - y + z = 2$ intersect in a line.

Solution:
Their intersection is empty because their normal vectors are the same and the equations are not just a scalar multiple of each other.

6) **T** F $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = 0$ implies $|u| = |v|$.

Solution:
Multiply out using the distributive law. The mixed terms cancel and we see $|\vec{u}|^2 = |\vec{v}|^2$.

7) | T | F | The contour curves $\sin(x) + y = 1$ and $\sin(x) + y = 2$ do not intersect.

Solution:
If these curves would intersect in a point (x, y), then f would take two values 1 and 2 at the same point, which is not possible. There are functions for which the contour lines intersect like $f(x, y) = x^2 - y^2/(x^2 + y^2$ but then this function is not continuous at 0.

8) | T | F | There is a vector $\vec{v}$ for which the vector projection $\text{proj}_{\vec{v}}(\vec{j})$ is equal to $2\vec{j}$.

Solution:
The vector projection is parallel to $\vec{v}$ not $\vec{j}$.

9) | T | F | $(\vec{k} \times \vec{i}) \times \vec{i} = \vec{j} \times (\vec{i} \times \vec{k})$

Solution:
The right hand side is zero, the left hand side not.

10) | T | F | If a curve $\vec{r}(t)$ lies in a plane, goes through the point $(1, 1, 1)$, and has the binormal vector $\vec{B}(t) = \langle 3, 4, 5 \rangle$, then the plane is $3x + 4y + 5z = 12$.

Solution:
The unit tangent and normal vector are in the plane. The binormal vector is perpendicular.

11) | T | F | The angle between $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$ is always 90 degrees.

Solution:
It is true for circles, but false in general. For example, on a line, the acceleration parallel to the velocity.

12) | T | F | A line intersects a hyperbolic paraboloid always in 2 distinct points.

Solution:
It can intersect in 1 point. Just take the z-axes for example.

13) | T | F | There is a quadric surface, each of whose intersections with the coordinate planes is either an ellipse or a parabola.

Solution:
The elliptic paraboloid does the job.

14) | T | F | The equation $x^2 - y^2 - z^2 = 1$ defines a one-sheeted hyperboloid.

Solution:
$f(x, y, z) = x^2 + y^2 - z^2 = 1$ is a one-sheeted hyperboloid

15) | T | F | The function $f(x, y) = 1/(1 + x^2 + y^2)$ is continuous everywhere.

Solution:
Take the derivatives.

16) | T | F | If the number $\vec{u} \cdot (\vec{v} \times \vec{w})$ is positive, then $(\vec{w} \times \vec{v}) \cdot \vec{u}$ is positive.

Solution:
Use that the cross product is anti commutative, and the dot product is commutative. You can find counter examples with simple choices like $\vec{u} = \vec{k}, \vec{v} = \vec{i}, \vec{w} = \vec{j}$.

17) | T | F | The number $|\vec{u} \times (\vec{v} \times \vec{w})|$ is the volume of the parallelepiped spanned by $\vec{u}, \vec{v}$ and $\vec{w}$.

Solution:
The volume is the triple scalar product not the triple crossed product.

18) T F The set of points P for which the distance of P to the point $(0,0,0)$ is 1 less than the distance to the point $(0,0,2)$ is a paraboloid.

Solution:
This was a homework problem. Remember the GPS problem? It is a hyperboloid.

19) T F If $\vec{v}, \vec{w}$ are two nonzero vectors, then the projection vector $\text{proj}_{\vec{w}}(\vec{v})$ can be longer than $\vec{v}$.

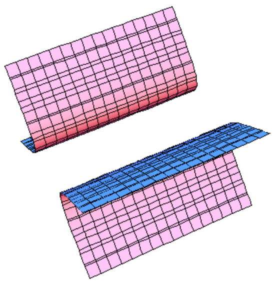
Solution:
The projection vector has length $|\vec{v} \cdot \vec{w}|/|\vec{w}|$ which has length smaller or equal to $\vec{v}$ (use the cos formula).

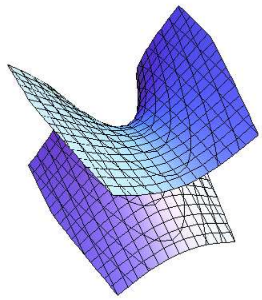
20) T F The number $|\vec{v} \times \vec{w}|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

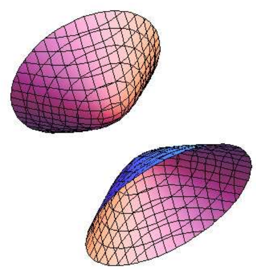
Solution:
Thats an important fact.

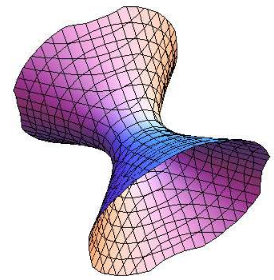
Problem 2a) (5 points)

Match the equations with the pictures. No justifications are necessary in this problem.

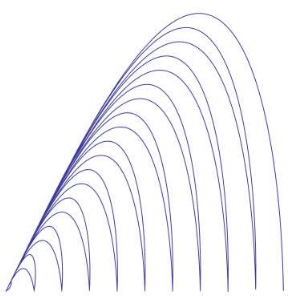

I

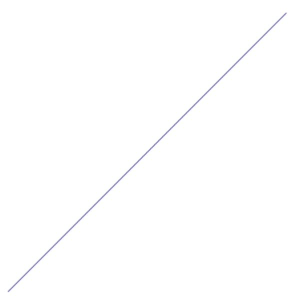

II

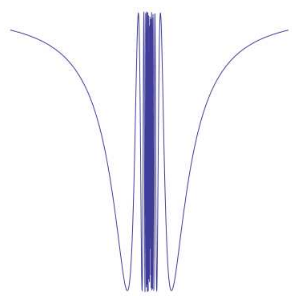

III

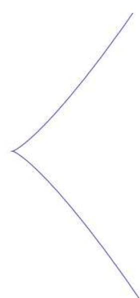

IV

Enter I,II,III,IV here	Equation
	$x^2 + y - z^2 - 1 = 0$
	$y^2 - 2z^2 - 1 = 0$
	$x^2 - y^2 + z^2 + 1 = 0$
	$x^2 - y^2 + z^2 - 1 = 0$

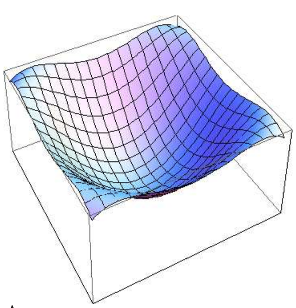

1

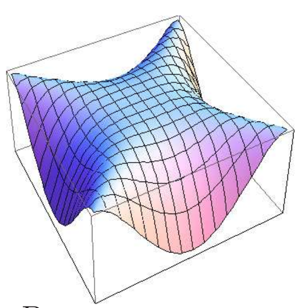

2

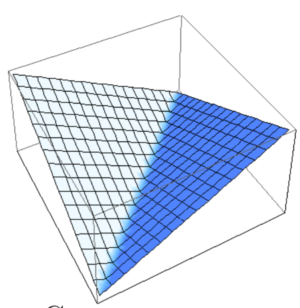

3

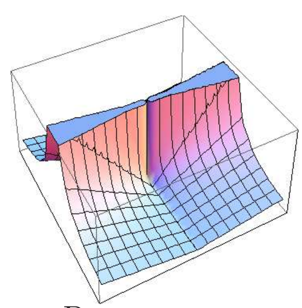

4

Enter 1,2,3,4 here	Equation
	$\langle t^4, 1 + t^5 \rangle$
	$\langle t\cos(5t), t\cos(5t) \rangle$
	$\langle \lvert t\cos(5t)\rvert, \lvert t\sin(10t)\rvert \rangle$
	$\langle 3 + 2t, \cos(1/t) \rangle$


A


B


C

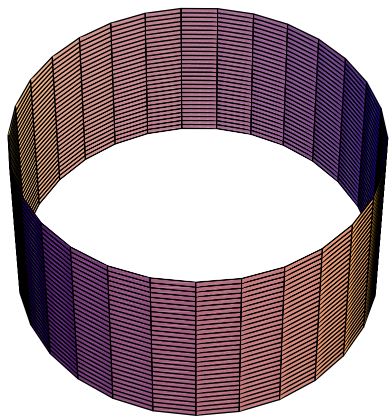

D

Enter A,B,C,D here	Equation
	$f(x, y) = \lvert x/y\rvert$
	$f(x, y) = \sin(x^2 + y^2)$
	$f(x, y) = \lvert x - y\rvert$
	$f(x, y) = \cos(x^2 - y^2)$

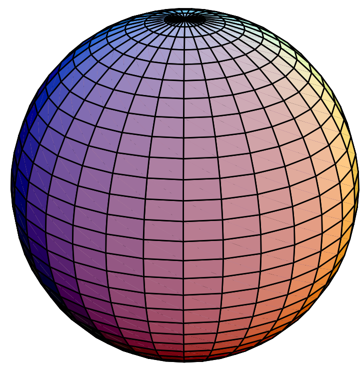
Solution:
II,I,III,IV, 4,2,1,3, D,A,C,B

Problem 2b) (5 points)

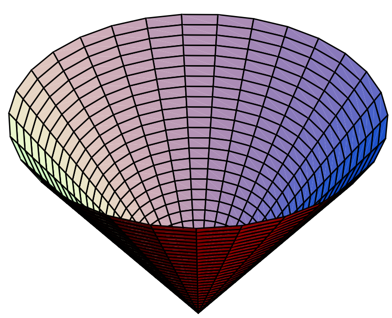
Match the surfaces with their parametrizations as well as with the description either in cylindrical coordinates (r, θ, z) or in spherical coordinates (ρ, ϕ, θ).



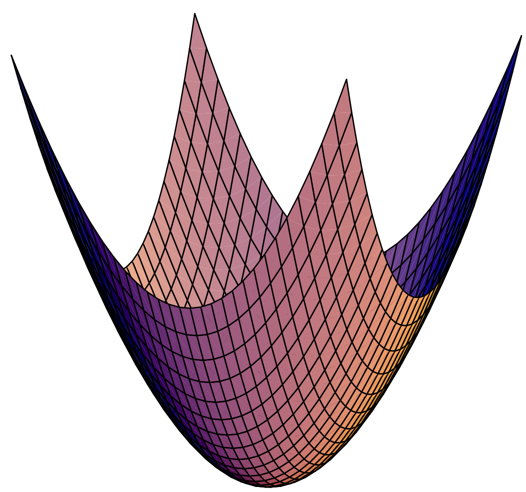
I



II



III



IV

Enter I,II,III,IV here	Parametrization of the surface
	$\langle 3\cos(\theta), 3\sin(\theta), 2z\rangle$.
	$\langle x, y, 3x^2 + 3y^2\rangle$
	$\langle 3\sin(\phi)\cos(\theta), 3\sin(\phi)\sin(\theta), 3\cos(\phi)\rangle$
	$\langle 3z\cos(\theta), 3z\sin(\theta), 2z\rangle$

Enter I,II,III,IV here	Description in cylindrical or spherical coordinates
	$z = 3r^2$
	$3z = 2r$
	$\rho = 3$
	$r = 3$

Solution:

Enter I,II,III,IV here	Parametrization of the surface
I	$\langle 3\cos(\theta), 3\sin(\theta), 2z\rangle$.
IV	$\langle x, y, 3x^2 + 3y^2\rangle$
II	$\langle 3\sin(\phi)\cos(\theta), 3\sin(\phi)\sin(\theta), 3\cos(\phi)\rangle$
III	$\langle 3z\cos(\theta), 3z\sin(\theta), 2z\rangle$

Enter I,II,III,IV here	Description in cylindrical or spherical coordinates
IV	$z = 3r^2$
III	$3z = 2r$
II	$\rho = 3$
I	$r = 3$

Problem 3) (10 points)

A tetrahedron has the vertices $A = (1, 1, 0), B = (3, 2, 0), C = (2, 1, 1), D = (3, 2, 1)$ with base triangle A, B, C.

a) (5 points) Find the height of the tetrahedron.

b) (5 points) The volume of a tetrahedron is the base area times height divided by 3. What is the volume of the tetrahedron with vertices A,B,C,D.

Solution:
a) The height is the distance of the point D to the plane spanned by $\vec{AB} = \langle 2, 1, 0\rangle$ and $\vec{AC} = \langle 1, 0, 1\rangle$. The normal vector $\vec{n} = \vec{AB} \times \vec{AC}$ is $\langle 1, -2, -1\rangle$. The distance is the scalare projection of $\vec{AD} = \langle 2, 1, 1\rangle$ onto $\vec{n}$ which is

$$d = \frac{\vec{n} \cdot \vec{AD}}{|\vec{n}|} = 1/\sqrt{6} \; .$$

b) The area of the base is the length of the vector $\vec{n}$ because the crossed product gives the area of a parallelepiped. The area is $\sqrt{6}/2$ because the triangle has half the area. The volume of the parallelepiped the height times the base area divided by 3 which is $\boxed{1/6}$.

Problem 4) (10 points)

Find the symmetric equation

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

of the intersection of the two planes

$$2x + y + z = 4$$

and

$$x - y + 2z = 5 \ .$$

Solution:
We can find the direction of the intersection by taking the cross product of $\langle 2,1,1\rangle$ with $\langle 1,-1,2\rangle$ which is $\langle 3,-3,-3\rangle$. One point of intersection is $(3,-2,0)$. We have therefore the parametrization $\vec{r}(t) = \langle 3,-2,0\rangle + t\langle 3,-3,-3\rangle = \langle 3+3t, -2-3t, -3t\rangle$. The symmetric equation is

$$\frac{x-3}{3} = \frac{y+2}{-3} = \frac{z}{-3} \ .$$

An other solution can be found by finding two intersection points P, Q and getting the direction of the intersection as $\vec{PQ}$. Note that there are many solutions to this problem, depending on which point on the line was.

Problem 5) (10 points)

What is the distance between the two cylinders $x^2 + y^2 = 1$ and $(z-2)^2 + (x-5)^2 = 4$?

Solution:
The parametrization of the first cylinder axis is $\vec{r}(t) = \langle 0,0,0\rangle + t\langle 0,0,1\rangle$. The parametrization of the second cylinder axis is $\vec{r}(t) = \langle 5,0,2\rangle + t\langle 0,1,0\rangle$. The distance is $(\langle 5,0,2\rangle - \langle 0,0,0\rangle) \cdot (\langle 0,0,1\rangle \times \langle 0,1,0\rangle / |\langle 1,0,0\rangle| = \langle 5,0,2\rangle \cdot \langle 1,0,0\rangle = 5$. The distance between the cylinders is $5 - 1 - 2 = 2$. The final answer is $\boxed{2}$.

Problem 6) (10 points)

Find the arc length of the parameterized curve

$$\vec{r}(t) = \langle 2\sin(t), \frac{t^4}{4} + \frac{1}{2t^2}, 2\cos(t)\rangle$$

from $t = 1$ to $t = 2$.

Solution:
The velocity is $\vec{r}\,'(t) = \langle 2\cos(t), t^3 + t^{-3} - 2\sin(t)\rangle$. The speed is $\sqrt{2 + t^3, +1/t^3} = (t^3 + 1/t^3)$. The integral

$$L = \int_1^2 (t^3 + 1/t^3)\; dt$$

gives $t^4/4 - 1/(2t^2|_1^2 = \boxed{33/8}$.

Problem 7) (10 points)

At time $t = 0$ two trapeze artists have positions $\vec{r}(0) = \langle 0, 0, 25\rangle$ and $\vec{s}(0) = \langle 10, 0, 23\rangle$ and velocities $\vec{r}\,'(0) = \langle 2, 0, 1\rangle$ and $\vec{s}\,'(0) = \langle -3, 0, 2\rangle$. They both experience a constant gravitational acceleration $\langle 0, 0, -10\rangle$. Find the paths $\vec{r}(t)$, $\vec{s}(t)$ and determine at which point the artists meet.

Solution:
This is a typical free fall problem. The main difficulty in this problem is to imagine, what happens when the two artists collide. The two artists actually were shot by cannons but we didn't want to scare you...
To the solution: Integrate $\vec{r}\,''(t)$ twice.

$$\vec{r}(t) = \langle 2t, 0, 25 + t - 5t^2\rangle \; .$$

$$\vec{s}(t) = \langle 10 - 3t, 0, 23 + 2t - 5t^2\rangle \; .$$

In order to find the intersection, we solve $\vec{r}(t) = \vec{s}(t)$. We can solve for t in any of the three coordinates. The artists meet at $t = 2$ at the point $\boxed{\langle 4, 0, 7\rangle}$.

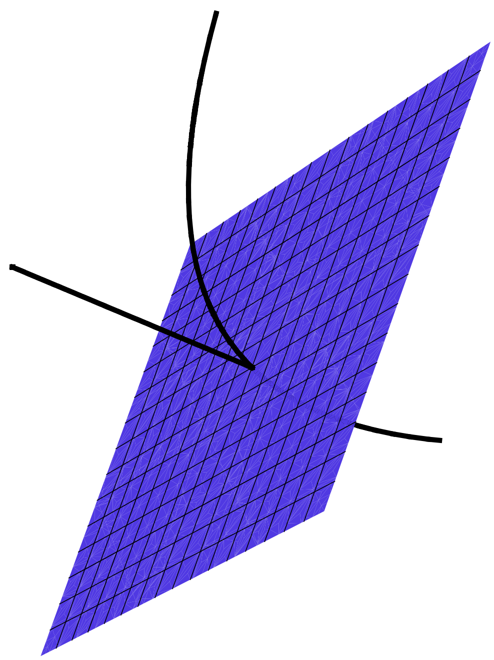
Problem 8) (10 points)

The angle between a curve and a plane is defined as $\pi/2 - \alpha$, where α is the angle between the normal vector to the plane and the velocity vector of the curve at the point of intersection.

a) (3 points) Find a normal vector to the plane $x + y - z/2 = 1$.

b) (3 points) What is the velocity vector to the curve $C : \vec{r}(t) = \langle 1, 0, 0\rangle + t\langle 1, 1, 3\rangle + t^2\langle 1, 1, 1\rangle$ at time $t = 0$?

c) (4 points) Find the angle (in radians) between the plane $x + y - z/2 = 1$ and the curve C at the point of intersection $\vec{r}(0)$.



Solution:
a) The normal vector to the plane is $\vec{n} = \langle 1, 1, -1/2\rangle$.
b) The velocity vector is $\vec{r}\,'(t) = \langle 1, 1, 3\rangle$.
c) We use the cos-formula: $\cos(\alpha) = \vec{r}\,'(t) \cdot \vec{n}/|\vec{r}\,'(t)||\vec{n}|$ and solve for

$$\cos(\alpha) = \frac{|\langle 1, 1, 3\rangle \cdot \langle 1, 1, -1/2\rangle|}{||\langle 1, 1, 3\rangle| \cdot |\langle 1, 1, -1/2\rangle|} = 1/(3\sqrt{11}\ .$$

Therefore, $\alpha = \arccos(1/3\sqrt{11})$ and the final answer is $\boxed{\pi/2 - \arccos(\frac{1}{3\sqrt{11}})}$.

Problem 9) (10 points)

The intersection of the paraboloid

$$x^2 + y^2 - z = 5$$

with the plane

$$x + y = 5$$

is a curve. Find the parametrization of this curve.

Solution:
From the second equation we can get $x = t$, $y = 5 - t$. From the first equation, we get $z = -5 + t^2 + (5 - t)^2$ which gives

$$\vec{r}(t) = \langle t, 5 - t, -5 + t^2 + (5 - t)^2 \rangle \ .$$

Problem 10) (10 points)

a) (3 points) Parametrize the plane containing the three points $A = (1, 1, 1)$, $B = (1, 3, 2)$ and $C = (3, 4, 5)$.
b) (4 points) Parametrize the sphere which is centered at $(1, 1, 1)$ and has radius 3.
c) (3 points) Parametrize the surface which is given in spherical coordinates as $\rho = 3 + \sin(\phi)\sin(\theta)$.

Solution:
a) $\vec{r}(s, t) = \langle 1, 1, 1 \rangle + t\langle 0, 2, 1 \rangle + s\langle 2, 3, 4 \rangle$.
b) $\vec{r}(\theta, \phi) = \langle 1 + 3\cos(\theta)\sin(\phi), 1 + 3\sin(\theta)\sin(\phi), 1 + 3\cos(\phi) \rangle$.
c) $\vec{r}(\theta, \phi) = (3 + \sin(\phi)\sin(\theta))\langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) \rangle$.