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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work.** Except for problems 1-3, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False (TF) questions (20 points)
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Mark for each of the 20 questions the correct letter. No justifications are needed.

- 1)

T	F
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 The surface $x^2 + y^2 + z^2 + 2z = 0$ is a sphere.

Solution:

Complete the square: add 1 on both sides, to get $x^2 + y^2 + (z + 1)^2 = 1$ which is a sphere.

- 2)

T	F
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 The length of the vector $\langle 1, 2, 2 \rangle$ is an integer.

Solution:

Indeed, $1 + 4 + 4 = 9$ is a perfect square.

- 3)

T	F
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 The vector $\langle 3, 4 \rangle$ appears as a velocity vector of the curve $\vec{r}(t) = \langle \cos(5t), \sin(5t) \rangle$. Namely, there is a t such that $\vec{r}'(t) = \langle 3, 4 \rangle$.

Solution:

The velocity vector of that curve has length 5 and takes any possible direction, so also the direction of the vector $\langle 3, 4 \rangle$.

- 4)

T	F
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 If $\vec{T}$ is the unit tangent vector, $\vec{N}$ is the unit normal vector, and $\vec{B}$ is the binormal vector, then $\vec{B} \times \vec{N} = \vec{T}$.

Solution:

The sign is wrong.

- 5)

T	F
---	---

 The curvature of a larger circle $r = 2$ is greater than the curvature of a smaller circle $r = 1/2$.

Solution:

False. The curvature of a circle of radius r is $1/r$.

- 6)

T	F
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 The surface $x^2 - y^2 - z^2 - 1 = 0$ is a one sheeted hyperboloid.

Solution:

It is a two sheeted one. Make a completion of a square

- 7)

T

F

 The function $f(x, y) = y^2 - x^2$ has a graph that is an elliptic paraboloid.

Solution:

It is a hyperbolic paraboloid.

- 8)

T

F

 Let $\vec{r}(t)$ be a parametrization of a curve. If $\vec{r}(t)$ is always parallel to the tangent vector $\vec{r}'(t)$, then the curve is part of a line through the origin.

Solution:

$\vec{r}'$ is zero at all times, so since $\vec{r}'(t)$ is parallel to the position vector $\vec{r}(t)$, the curve must lie on a line through the origin.

- 9)

T

F

 If $\text{Proj}_{\vec{k}}(\vec{u})$ is perpendicular to $\vec{u}$, then $\vec{u}$ is the zero vector.

Solution:

Take $\vec{u} = \vec{i}$.

- 10)

T

F

 If $\text{Proj}_{\vec{k}}(\vec{u})$ is perpendicular to $\vec{u}$, then $\text{Proj}_{\vec{k}}(\vec{u})$ is the zero vector.

Solution:

Assume $\text{Proj}_{\vec{k}}(\vec{u})$ is a nonzero vector in the direction of $\vec{k}$. This means that $\vec{k}$ is perpendicular to $\vec{u}$, so $\text{Proj}_{\vec{k}}(\vec{u}) = \vec{0}$ by the definition of the projection.

- 11)

T

F

 If $\vec{u} \times \vec{v} = \vec{0}$ then $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$.

Solution:

The two vectors can be parallel.

- 12)

T	F
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 There are two vectors $\vec{a}$ and $\vec{b}$ such that the scalar projection of $\vec{a}$ onto $\vec{b}$ is 100 times the magnitude of $\vec{b}$.

Solution:

Take $\vec{b} = 100\vec{a}$.

- 13)

T	F
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 The curve $\vec{r}(t) = \langle \cos(t), e^t + 10, t^2 \rangle, 2 \leq t \leq 6$ and the curve $\vec{r}(t) = \langle \cos(2t), e^{2t}, 4t^2 \rangle, 1 \leq t \leq 3$ have the same length.

Solution:

Make a change of variables.

- 14)

T	F
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 The equation $\rho \sin(\phi) - 2 \sin(\theta) = 0$ in spherical coordinates defines a two sheeted hyperboloid.

Solution:

The equation means $r = 2 \sin(\theta)$ or $x^2 + y^2 = 2y$.

- 15)

T	F
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 If triple scalar product of three vectors $\vec{u}, \vec{v}, \vec{w}$ is larger than $|\vec{u} \times \vec{v}|$ then $|\vec{w}| > 1$.

Solution:

Think of the triple scalar product as the volume. If that is larger than the base area, the height has to be bigger than 1.

- 16)

T	F
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 The distance between the x -axis and the line $x = y = 1$ is $\sqrt{2}$.

Solution:

The distance is 1. The distance between $x = y = 1$ and the z -axis would be $\sqrt{2}$.

- 17)

T	F
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 The vector $\langle -1, 2, 3 \rangle$ is perpendicular to the plane $x - 2y - 3z = 9$.

Solution:

It is. Because $\langle 1, -2, -3 \rangle$ is perpendicular to the plane.

- 18)

T	F
---	---

 The curve $\vec{r}(t) = t^3\langle 1, 2, 3 \rangle$ is a line.

Solution:

Even so t appears not in a linear way, the curve is still a line.

- 19)

T	F
---	---

 The point $(1, 1, -\sqrt{3})$ is in spherical coordinates given by $(\rho, \theta, \phi) = (\sqrt{5}, \pi/4, 2\pi/3)$.

Solution:

Also the z coordinate is wrong.

- 20)

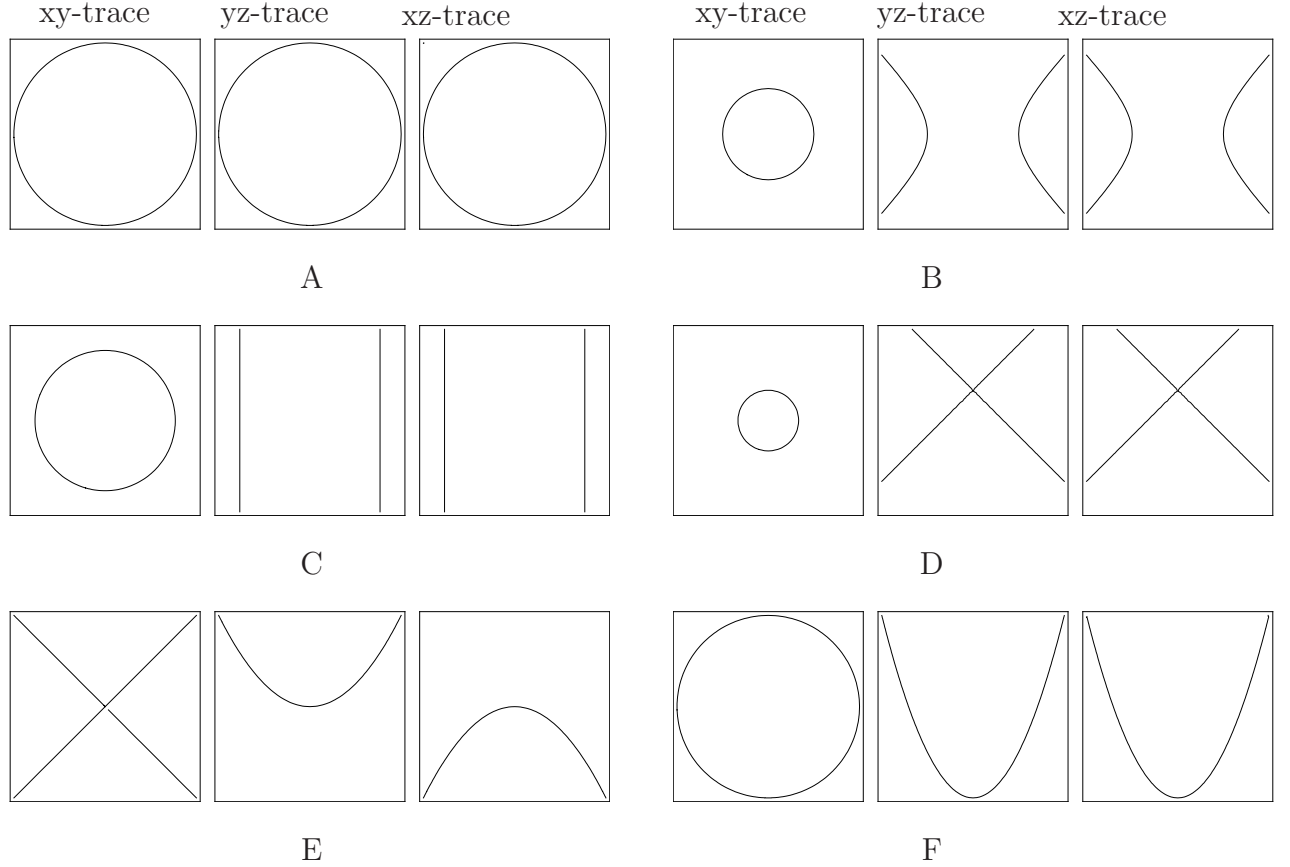
T	F
---	---

 If the cross product satisfies $(\vec{v} \times \vec{w}) \times \vec{v} = \vec{0}$ then $\vec{v}$ and $\vec{w}$ are orthogonal.

Solution:

One can have $\vec{v} = \vec{w}$ in which case the two vectors are not orthogonal and still the product in question is zero.

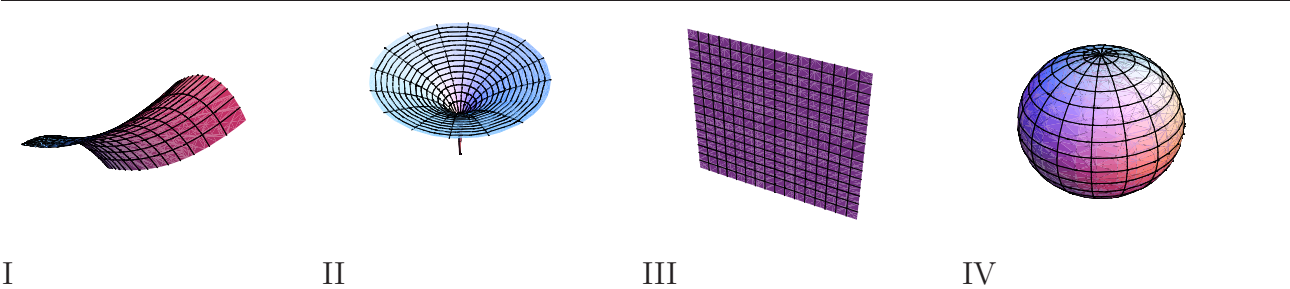
Problem 2a) (6 points)



The figures above show the xy-trace, (the intersection of the surface with the xy-plane), the yz-trace (the intersection of the surface with the yz-plane), and the xz-trace (the intersection of the surface with the xz-plane). Match the following equations with the traces. No justifications required.

Enter A,B,C,D,E,F here	Equation
	$x^2 + y^2 - (z - 1/3)^2 = 0$
	$x^2 + y^2 + z^2 - 1 = 0$
	$x^2 - y^2 - z = 0$
	$x^2 + y^2 - 1 = 0$
	$x^2 + y^2 - z^2 - 1 = 0$
	$x^2 + y^2 - z = 1$

Problem 2b) (4 points)



Match the parametric surfaces with their parameterization. No justifications are needed.

Enter I,II,III,IV here	Parameterization
	$\vec{r}(u, v) = \langle u^2, v^2, u^4 - v^4 \rangle$
	$\vec{r}(u, v) = \langle \cos(u) \sin(v), 1 + \sin(u) \sin(v), \cos(v) \rangle$
	$\vec{r}(u, v) = \langle v \cos(u), v \sin(u), v^{1/4} \rangle$
	$\vec{r}(u, v) = \langle u, 3, v \rangle$

Solution:

a) DAECBF.

b) I,IV,II,III

Problem 3) (10 points)

Find the distance of the point $P = (3, 4, 5)$ to the line

$$\frac{x-1}{4} = \frac{y-2}{5} = \frac{z-3}{6}.$$

Solution:

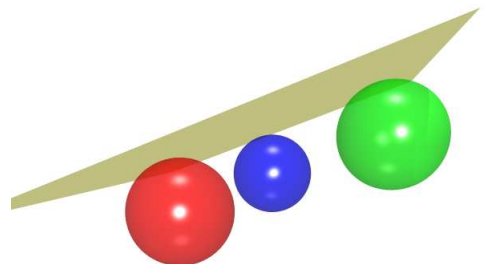
First parametrize the line as $\vec{r}(t) = \langle 1, 2, 3 \rangle + t\langle 4, 5, 6 \rangle = \langle 1, 2, 3 \rangle + t\vec{v}$. We also need the vector $\vec{QP} = \langle 2, 2, 2 \rangle$ connecting a point $Q = (1, 2, 3)$ on the line with the point $P = (3, 4, 5)$. Use the distance formula

$$d = |\langle 4, 5, 6 \rangle \times \langle 2, 2, 2 \rangle| / |\langle 4, 5, 6 \rangle| = |\langle -2, 4, -2 \rangle| / |\langle 4, 5, 6 \rangle|$$

$$= \sqrt{24/77}.$$

Problem 4) (10 points)

Given three spheres of radius 1 centered at $A = (1, 2, 0)$, $B = (4, 5, 0)$, $C = (1, 3, 2)$. Find a plane $ax + by + cz = d$ which touches each of three spheres from the same side.



Solution:

The normal vector to the plane is $\vec{n} = \langle 3, 3, 0 \rangle \times \langle 0, 1, 2 \rangle = \langle 6, -6, 3 \rangle$. The plane touching the three spheres has the equation $6x - 6y + 3z = d$, where d is a constant still to be determined. To find this constant, we have to find a point P on the plane. We do that by going from the point A by 1 unit in the direction of the normal vector. The point $P = A + \vec{n}/|\vec{n}| = (1, 2, 0) + \langle 6/9, -6/9, 3/9 \rangle = (5, 4, 1)/3$ is on the plane. Plug in this point into the equation gives $d = 3$. The equation of the plane is $\boxed{6x - 6y + 3z - 3 = 0}$.

Problem 5) (10 points)

Find the arc length of the curve

$$\vec{r}(t) = \langle t^3/3, t^4/2, 2t^5/5 \rangle$$

from $0 \leq t \leq 1$.

Solution:

We have $\vec{r}'(t) = \langle t^2, 2t^3, 2t^4 \rangle$ and $|\vec{r}'(t)| = \sqrt{t^4 + 4t^6 + 4t^8} = t^2 + 2t^4$. The arc length is

$$\int_0^1 t^2 + 2t^4 \, dt = 1/3 + 2/5 = 11/15.$$

The arc length is $\boxed{11/15}$.

Problem 6) (10 points)

An apple at position $(0, 0, 20)$ rests 20 meters above Newton's head, the tip of whose nose is at $(1, 0, 0)$. The apple falls with constant acceleration $\vec{r}''(t) = \langle a, 0, -10 \rangle$ (where $\langle 0, 0, -10 \rangle$ is caused by gravity and $\langle a, 0, 0 \rangle$ by the wind) precisely onto the nose of Newton. Find the wind force $\langle a, 0, 0 \rangle$ which achieves this. Give a parametrization for the path along which the apple falls.



Solution:

From

$$\vec{r}''(t) = \langle a, 0, -10 \rangle$$

we get by integration

$$\vec{r}'(t) = \langle at, 0, -10t \rangle + \langle 0, 0, 0 \rangle$$

and

$$\vec{r}(t) = \langle at^2/2, 0, -5t^2 \rangle + \langle 0, 0, 20 \rangle = \langle at^2/2, 0, -5t^2 + 20 \rangle .$$

Now, in order that we reach the nose $(1, 0, 0)$, we have to get the time t such the apple is at the ground $5t^2 = 20$ gives $t = 2$. In order that $at^2/2 = 1$ we have $a = 1/2$. The wind force is $\langle 1/2, 0, 0 \rangle$. The path is

$$\vec{r}(t) = \langle t^2/4, 0, 20 - 5t^2 \rangle .$$

Problem 7) (10 points)

a) (5 points) A red maple leaf falls to the ground $z = 0$. It falls along the curve $\vec{r}(t) = \langle 3\sqrt{3}\cos(t), 3\sqrt{3}\sin(t), 5-t-4t^2 \rangle$. At which angle does it hit the xy -plane?

b) (5 points) Find the tangent line to the curve at the impact point.

**Solution:**

a) The leaf hits the ground at the point $\vec{r}(1) = \langle 3\sqrt{3}\cos(1), 3\sqrt{3}\sin(1), 0 \rangle$ at time $t = 1$. We compute the velocity vector at that time

$$\vec{r}'(1) = \langle -3\sqrt{3}\sin(1), 3\sqrt{3}\cos(1), -9 \rangle .$$

In order to compute the impact angle, we compute the angle with the normal vector $\langle 0, 0, 1 \rangle$ which is $\cos(\alpha) = -9/(6\sqrt{3}) = -\sqrt{3}/2$ so that $\alpha = 5\pi/6$. The angle between the plane and the velocity vector is $5\pi/6 - \pi/2 = \pi/3$. The result is $\boxed{\pi/3}$.

b) the tangent line has the parametrization $\vec{r}(t) = \vec{r}(1) + t\vec{r}'(1)$ which is

$$\vec{r}(t) = \langle 3\sqrt{3}\cos(1), 3\sqrt{3}\sin(1), 0 \rangle + t\langle -3\sqrt{3}\sin(1), 3\sqrt{3}\cos(1), -9 \rangle .$$

Problem 8) (10 points)

a) (5 points) The surface

$$\vec{r}(t, s) = \langle 1 + t + s, 1 - t - 2s, 1 + t - s \rangle$$

with $0 \leq t \leq 1, 0 \leq s \leq 1$ is a parallelogram in space. Find the area of this parallelogram.

b) (5 points) Another surface is given in spherical coordinates by $\rho = 2 \sin(\phi) \cos(\theta)$. Write down the equation of this surface in rectangular coordinates as well as in cylindrical coordinates.

Solution:

a) The parallelogram is spanned by $\langle 1, -1, 1 \rangle$ and $\langle 1, -2, -1 \rangle$. The area is the length of the cross product $\langle 3, 2, -1 \rangle$ which is $\sqrt{14}$.

b) To get the equation in rectangular coordinates, multiply both sides of the equation with ρ , this gives

$$x^2 + y^2 + z^2 = \rho^2 = 2\rho \sin(\phi) \cos(\theta) = 2x .$$

Complete the square to get $(x - 1)^2 + y^2 + z^2 = 1$.

To get from rectangular to cylindrical coordinates, just replace $x^2 + y^2$ with r^2 and x with $r \cos(\theta)$ and leave z as it is. In cylindrical coordinates the surface is given by the equation

$$r^2 - 2r \cos(\theta) + z^2 = 0 .$$

Problem 9) (10 points)

a) (5 points) Parametrize the curve obtained by intersecting the surface $z - x^2 + y^3 = 0$ with the cylindrical surface $x^2/4 + 9y^2 = 1$.

b) (5 points) Find the unit tangent vector $\vec{T}$ and the normal vector $\vec{N}(t) = \vec{T}'(t)/|\vec{T}'(t)|$ to the curve

$$\vec{r}(t) = \langle 3, t^2, t \rangle$$

at the point $(3, 0, 0)$. What is the binormal vector $\vec{B} = \vec{T} \times \vec{N}$?

Solution:

a) First parametrize the first two coordinates: $\vec{r}(t) = \langle 2 \cos(t), 1/3 \sin(t), \dots \rangle$, then fill in the third coordinate $z = x^2 - y^3$ to get

$$\vec{r}(t) = \langle 2 \cos(t), 1/3 \sin(t), 4 \cos^2(t) - \sin^3(t)/27 \rangle$$

b) $\vec{T}(t) = \langle 0, 2t, 1 \rangle / \sqrt{1 + 4t^2}$. At $t = 0$ we have $\vec{T} = \langle 0, 0, 1 \rangle$. Now $\vec{T}'(t) = \langle 0, 2, 0 \rangle / \sqrt{1 + 4t^2} - 4t \langle 0, 2t, 1 \rangle / (1 + 4t^2)^{3/2}$. Which is at $t = 0$ equal to $\langle 0, 2, 0 \rangle$. Normalized, we get $\vec{N} = \langle 0, 1, 0 \rangle$. The third vector is just the cross product of the first two $\vec{B} = \langle -1, 0, 0 \rangle$.

Problem 10) (10 points)

- a) (4 points) Give a parametrization of the hyperboloid $x^2 + y^2 = z^2 + 1$.
- b) (3 points) Give a parametrization of the plane $x + y = 1$.
- c) (3 points) Give a parametrization of the ellipsoid $x^2 + y^2 + z^2/4 = 1$.

Solution:

a) Since the hyperboloid is in cylindrical coordinates given as $r^2 = z^2 + 1$, we have

$$\vec{r}(z, \theta) = \langle \sqrt{z^2 + 1} \cos(\theta), \sqrt{z^2 + 1} \sin(\theta), z \rangle .$$

b) Since the points $(1, 0, 0), (0, 1, 0), (1, 0, 1)$ are in the plane, we have two vectors $\vec{v} = \langle -1, 1, 0 \rangle, \vec{w} = \langle 0, 0, 1 \rangle$ parallel to the plane and can parametrize

$$\vec{r}(s, t) = \langle 1, 0, 0 \rangle + t \langle -1, 1, 0 \rangle + s \langle 0, 0, 1 \rangle .$$

c) The ellipsoid is a deformation of the sphere. We just have to make it higher:

$$\vec{r}(\theta, \phi) = \langle \sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), 2 \cos(\phi) \rangle .$$