KEPLER'S LAWS

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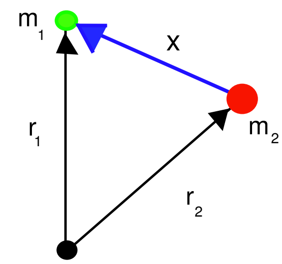
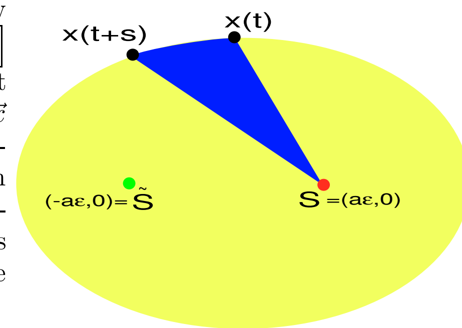
FROM THE 2 BODY TO THE KEPLER PROBLEM. Two bodies with position vectors $\vec{r}_1, \vec{r}_2$ and mass m_1, m_2 attract each other with the force $\vec{F} = -Gm_1m_2\frac{\vec{x}}{r^3}$, where $\vec{x} = \vec{r}_1 - \vec{r}_2$ is the difference vector between the bodies. The length $|\vec{F}|$ of the force is proportional to $1/r^2$. The vector $\vec{x} = \vec{r}_1 - \vec{r}_2$ of length r satisfies $\vec{x}'' = \vec{r}_1'' - \vec{r}_2'' = m_2G(\vec{r}_2 - \vec{r}_1)/|\vec{r}_2 - \vec{r}_1|^3 - m_1G(\vec{r}_1 - \vec{r}_2)/|\vec{r}_2 - \vec{r}_1|^3 = -(m_1 + m_2)G\vec{x}/r^3$. The new problem describes a mass-point with position $\vec{x}$ and mass $m = m_1 + m_2$ which is attracted to a fixed force centered at the origin. This problem $\boxed{\vec{x}'' = -mG\vec{x}/r^3}$ is called the **Kepler problem**. The **angular momentum** $\vec{L} = m\vec{x} \times \vec{x}'$ and the **energy** $E = m|\vec{x}'|^2/2 + mG/|\vec{x}|$ do not change in time.

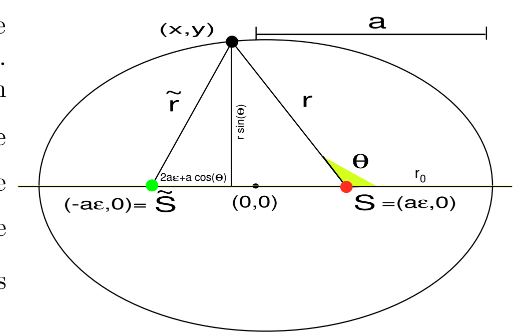


THE 2. KEPLER LAW. The "area law" or Keplers second law is: "The radius vector $\vec{x}$ passes the equal area in equal time."

Proof. Because $\vec{x}''$ is parallel to $\vec{x}$, and $L = m\vec{x}' \times \vec{x}$, we get $\vec{L}' = 0$ (use the product law). If follows that the vector $\vec{x}$ stays in the plane spanned by $\vec{x}$ and $\vec{x}'$. We can now use coordinates $\vec{x} = (r\cos(\theta), r\sin(\theta), 0)$ to describe the point. With $\vec{x}' = (r'\cos(\theta), r'\sin(\theta), 0) + (-r\sin(\theta), r\cos(\theta), 0)\theta'$. The conserved quantity $|L| = m|\vec{x}' \times \vec{x}| = mr^2\theta'$ can be interpreted as $2mf'$, where $f(t)$ is the **area** swept over by the vector x in the interval $[0, t]$. We will use the formula $\boxed{|L| = r^2\theta'}$ later.



ELLIPSES. An **ellipse** with **focal points** $\tilde{S} = (-a\epsilon, 0), S = (a\epsilon, 0)$ with **eccentricity** ϵ is defined as the set of points in the plane whose distances $\tilde{r}$ and r to $\tilde{S}$ and S satisfy $\tilde{r} + r = 2a$. From $(2a - r)^2 = \tilde{r}^2 = r^2\sin^2(\theta) + (2a\epsilon + r\cos(\theta))^2$, we obtain $\boxed{r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta)}}$ the **polar form** of the ellipse. One can replace a with the constant $r_0 = (1 - \epsilon)a$, the length of $\vec{r}$ at $\theta = 0$. The polar form becomes then $\boxed{r = \frac{r_0(1+\epsilon)}{1+\epsilon\cos(\theta)}}$. The **semiaxes** of the ellipse have length a and $b = a\sqrt{1 + \epsilon^2}$. The **area** is $\boxed{A = \pi ab}$ as we will see later in the course.



THE 1. KEPLER LAW. The radius vector $\vec{x}(t)$ describes an ellipse.

Proof. (We show that a particle on an ellipse satisfying the 2. law has the correct acceleration of the Kepler law). From the 2. law, we know $\theta' = L/(mr^2)$. The polar form allows us to find the time derivatives

$$r' = \frac{a(1-\epsilon^2)\sin(\theta)}{(1+\epsilon\cos(\theta))^2}\frac{L}{mr^2} = \frac{L\epsilon\sin(\theta)}{ma(1-\epsilon^2)}$$

of $r(t)$. (In the last step, we replaced r^2 in the denominator with the polar formula.) The second derivative is $\boxed{r'' = \frac{L^2\epsilon\cos(\theta)}{(m^2a(1-\epsilon^2)r^2)}}$ (1).

With the unit vector $\vec{n} = \vec{x}/r$, one has $\vec{x}'' = (\vec{x}'' \cdot \vec{n})\vec{n}$. From $\vec{x} = \vec{n}r$, we get $\vec{x}' = \vec{n}'r + \vec{n}r'$, $\vec{x}'' = \vec{n}''r + 2\vec{n}'r' + \vec{n}r''$. Therefore $\boxed{\vec{x}'' \cdot \vec{n} = \vec{n}'' \cdot \vec{n}r + 2\vec{n}' \cdot \vec{n}r' + r''}$ (2).

From $\vec{n} = (\cos(\theta), \sin(\theta), 0)$, we have $\vec{n}' = \theta'\vec{n}^\perp$ where $\vec{n}^\perp = (-\sin(\theta), \cos(\theta), 0)$. This gives $\vec{n}' \cdot \vec{n}' = (\theta')^2$. Using $\vec{n} \cdot \vec{n} = 1$, we have $\boxed{\vec{n} \cdot \vec{n}' = 0}$ (3). From $\vec{n}'' \cdot \vec{n} + \vec{n}' \cdot \vec{n}' = 0$ and $\vec{n}' \cdot \vec{n}' = (\theta')^2 = L^2/(mr^2)^2$ we obtain $\boxed{\vec{n}'' \cdot \vec{n}r = -\frac{L^2r}{(mr^2)^2} = -\frac{L^2}{m^2}\frac{(1+\epsilon\cos(\theta))}{r^2a(1-\epsilon^2)}}$ (4). Plugging (1),(3),(4) into (2) gives

$$\vec{x}'' \cdot \vec{n} = r'' + \vec{n}'' \cdot \vec{n}r = \frac{L^2\epsilon\cos(\theta)}{m^2a(1-\epsilon^2)r^2} - \frac{L^2}{m^2}\frac{(1+\epsilon\cos(\theta))}{r^2a(1-\epsilon^2)} = \frac{-L^2}{m^2a(1-\epsilon^2)r^2}.$$

Therefore,

$$\vec{x}'' = (\vec{x}'' \cdot \vec{n})\vec{n} = (\vec{x}'' \cdot \vec{n})\frac{\vec{x}}{r} = \frac{-L^2}{m^2a(1-\epsilon^2)}\frac{\vec{x}}{r^3} = -mG\frac{\vec{x}}{r^3}.$$

THE 3. KEPLER LAW. Let T be the **period** of the orbit. It is the time the body needs to go around the ellipse once. (If S is the sun and $r(t)$ the orbit of the earth, then T is one year.) The third Kepler law states that $\boxed{T^2/a^3 = 4\pi^2/(Gm) \text{ is a constant}}$.

Proof. If $f(t)$ is the area swept by the radial vector from time 0 to time t, then $f'(t) = L/(2m)$ implies that the area of the ellipse $A = \pi a^2\sqrt{1-\epsilon^2}$ is equal to $LT/(2m)$. From $T = 2m\pi a^2\sqrt{1-\epsilon^2}/L$, we obtain $T^2/a^3 = 4m^2\pi^2 a(1-\epsilon^2)/L^2 = 4\pi^2/(Gm)$.

MEASURING THE GRAVITATIONAL CONSTANT. Note that the 3. Kepler law allows us to compute the gravitational constant G from the period, the total mass (essentially the mass of the sun) and the geometry of the ellipse.

The case $\epsilon > 1$ corresponds to a negative G, where particles repell each other. The third Kepler law does then no more apply and ellipses become a hyperbola. The second law is unchanged.

REMARKS.

If the force is changed to $\vec{x}'' = -mG\vec{x}/r^\alpha$, (note that $\alpha = 3$ is the Kepler case), then the second Kepler law still applies, the other two laws not. The formula $\dot{\theta} = L/(mr^2)$ still applies. Also the derivation of the formula for $\vec{x}'' \cdot n = r'' - L^2/(m^2r^3)$ is valid. The left hand side is $-mG/r^{\alpha-1}$, which leads to the ordinary differential equation $r'' = -mG/r^{\alpha-1} + L^2/(m^2r^3)$ for $r(t)$. Knowing $r(t)$ gives then $\theta(t)$ from $\dot{\theta} = L/(mr^2)$. The global behavior depends on the constants G, L. If $\alpha = 4$, which corresponds to the two body problem in 4 dimensions, then $r'' = C/r^3$, where $C = -mG + L^2/m^2$ is a constant. If $C > 0$, the bodies seperate to infinity. If $C < 0$, then $r(t) \to 0$. Only if the angular momentum is such that $C = 0$, there is a bounded motion. **Stable planetary systems would not exist in four dimensions**. A theorem of Bertant states that only for $\alpha = 3$ (the Kepler case) and $\alpha = -1$ (the harmonic oscillator), all bounded orbits are periodic.

COLLISIONS.

If the two podies collide, we get a **collision singularity**. Collisions can occur in the 2-body problem, if the total angular momentum of the two bodies is zero. Analysing collision singularities involving more than two bodies helps to understand what happens when particles are close to such collision configurations. It is known that initial conditions leading to collisions are rare in the n-body problem. Noncollision singularities in which particles escape to infinity in finite time exist already for the 5-body problem.

Our galaxy and M31, the Andromeda galaxy, form a relatively isolated system known as the **local group**. The center of mass of M31 approaches the center of mass our galaxy with a velocity of 119 km/s. In about 10^{10} years, these galaxies are likely to collide. Such a collision will have dramatic consequences for both systems. Nevertheless, even a direct encounter would probably not lead to any collision of any two stars.

SOME HISTORY.

- Nicolas Kopernicus (1473-1543) had a heliocentric system.
- Galileo Galilei (1564-1642) discovers Jupiter moons.
- Johannes Kepler (1571-1630) builds on the observations of Tycho Brahe. Finds the first and second Kepler Law in 1609, the third in 1619.
- Before Newton, the dynamics of celestial objects was described empirically, first by circular, later by epicycle approximations which effectively where Fourier approximations of the actual elliptic motion.
- Issac Newton (1643-1727) led the foundations of mathematical description and developed calculus simultaneously with Leibnitz. Newton had already solved the Kepler problem geometrically.
- Johan Bernoulli proved in 1710 that solutions to the 2 body problem move on conic sections.
- Laplace, Lagrange, Hamilton and Poisson belong to the ancestors of celestial mechanics, the study of the n-body problem.
- With Poincaré (1854-1912) at the end of the 19'th century, and Birkhoff (at Harvard!), the subject of the three body problem was studied with new geometric and topological methods.