5/7/2004 CALCULUS BEYOND CALCULUS — Math21a, O. Knill

INTRODUCTION. Topics beyond multi-variable calculus are usually labeled with special names like "linear algebra", "ordinary differential equations", "numerical analysis", "partial differential equations", "functional analysis" or "complex analysis". Where one would draw the line between calculus and non-calculus topics is not clear but if calculus is about learning the basics of limits, differentiation, integration and summation, then multi-variable calculus is the "black belt" of calculus. Are there other ways to play this sport?

HOW WOULD ALIENS COMPUTE? On an other planet, calculus might be taught in a completely different way. The lack of fresh ideas in the current textbook offerings, where all books are essentially clones (*) of each other and where innovation is faked by ejecting new editions every year (which of course has the main purpose to prevent the retail of second hand books), one could think that in the rest of the universe, multi-variable calculus would have to be taught in the same way, and where chapter 12 is always the chapter about multiple integrals. Actually, as we want to show here, even the human species has come up with a wealth of different ways to deal with calculus. It is very likely that calculus textbooks would look very different in other parts of our galaxy. This week broke the news that one has discovered a 5'th arm of the milky way galaxy. It is 77'000 light years long and should increase the chance that there are other textbooks in our home galaxy. In this text, we want to give an idea that the calculus topics in this course could be extended or built completely differently. Actually, even **numbers** can be defined differently. John Conway introduced once numbers as pairs $\{L|R\}$ where L and R are sets of numbers defined previously. For example $\{\emptyset|\emptyset\} = 0$ and $\{0|\emptyset\} = 1$, $\{\emptyset|0\} = -1$, $\{0|1\}$. The advantage of this construction is that it allows to see "numbers" as part of "games". Donald E. Knuth, (the giant of a computer scientist, who also designed "TeX", a typesetting system in which this text is written), wrote a book called "surreal numbers" in which two students find themselves on an island. They find a stone with the axioms for a new number system is written and develop from that an entirely new number system which contains the real line and more. The book is a unique case, where mathematical discovery is described as a novel. This is so totally different from what we know traditionally about numbers that one could expect Conway to be an alien himself if there were not many other proofs of his ingenious creativity.

(*) One of the few exceptions is maybe the book of Marsden and Tromba, which is original, precise and well written. It is unfortunately slightly too mathematical for most calculus consumers and suffers from the same disease that other textbooks have: **a scandalous prize**. A definite counterexample is the book "how to ace calculus" which is funny, original and contains the essential stuff. And it comes as a **paperback**. Together with a Schaum outline volume (also in paperback), it would suffice as a rudimentary textbook combination (and would cost together half and weight one fourth) of the standard door stoppers.

NONSTANDARD CALCULUS. At the time of Leonard Euler, people thought about calculus in a more intuitive, but less formal way. For example $\boxed{(1+x/n)^n = e^x}$ with **infinitely large** n would be perfectly fine. A modern approach which catches this spirit is "nonstandard analysis", where the notion of "infinitesimal" is given a precise meaning. The simplest approach is to extend the language and introduce **infinitesimal** as objects which are smaller than all **standard** objects. We say $x \sim y$ if $|x-y|$ is infinitesimal. All numbers are traditionally defined like π or $\sqrt{2}$ are "standard". The notion which tells that every bounded sequence has an accumulation point is expressed by the fact that there exits a finite set A such that all $x \in D$ are infinitesimally close to an element in A. The fact that a continuous function on a compact set takes its maximum is seen by taking $M = \max_{a \in A} f(a)$. What impressed me as an undergraduate student learning nonstandard calculus (in a special course completely devoted to that subject) was the elegance of the language as well as the compactness in which the entire calculus story could be packed. For example, to express that a function f is continuous, one would say $x \sim y$ then $f(x) \sim f(y)$. This is more intuitive than the Weierstrass definition $\forall \epsilon > 0 \exists \delta > 0 |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$ which is understood today primarily by intimidation. To illustrate this, Ed Nelson, the founder of one of the nonstandard analysis flavors, asks the meaning of $\boxed{\forall \delta > 0 \exists \epsilon > 0 |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon}$ in order to demonstrate how unintuitive this definition really is. The derivative of a function is defined as the standard part of $f'(x) = (f(x+dx) - f(x))/dx$, where dx is infinitesimal. Differentiability means that this expression is finite and independent of the infinitesimal dx chosen. Integration $\int_a^x f(y)\, dy$ is defined as the standard part of $\sum_{x_j \in [a,x]} f(x_i) dx$ where $x_k = kdx$ and dx is an infinitesimal. The fundamental theorem of calculus is the triviality $F(x+dx) - F(x)) = f(x)dx$.

The reasons that prevented nonstandard calculus to go mainstream were lack of marketing, bad luck, being below a certain critical mass and the initial believe that students would have to know some of the foundations of mathematics to justify the game. It is also quite a sharp knife and its easy to cut the finger too and doing mistakes. The name "nonstandard calculus" certainly was not fortunate too. People call it now "infinitesimal calculus". Introducing the subject using names like "hyper-reals" and using "ultra-filters" certainly did not help to promote the ideas (we usually also don't teach calculus by introducing Dedekind cuts or completeness) but there are books like by Alain Robert which show that it is possible to teach nonstandard calculus in a natural way.

DISCRETE SPACE CALCULUS. Many ideas in calculus make sense in a discrete setup, where space is a graph, curves are curves in the graph and surfaces are collections of "plaquettes", polygons formed by edges of the graph. One can look at functions on this graph. Scalar functions are functions defined on the vertices of the graphs. Vector fields are functions defined on the edges, other vector fields are defined as functions defined on plaquettes. The gradient is a function defined on an edge as the difference between the values of f at the end points.

Consider a network modeled by a planar graph which forms triangles. A **scalar function** assigns a value f_n to each node n. An **area function** assigns values f_T to each triangle T. A **vector field** assign values F_{nm} to each edge connecting node n with node m. The **gradient** of a scalar function is the vector field $F_{nm} = f_n - f_m$. The **curl** of a vector field F is attaches to each triangle (k,m,n) the value $\text{curl}(F)_{kmn} = F_{km} + F_{mn} + F_{nk}$. It is a measure for the circulation of the field around a triangle. A curve γ in our discrete world is a set of points $r_j, j = 1, \dots, n$ such that nodes r_j and r_{j+1} are adjacent. For a vector field F and a curve γ, the **line integral** is $\sum_{j=1}^n F_{r(j)r_{j+1}}$. A **region** R in the plane is a collection of triangles T. The **double integral** of an area function f_T is $\sum_{T \in R} f_T$. The **boundary** of a region is the set of edges which are only shared by one triangle. The orientation of γ is as usual. Greens theorem is now almost trivial. Summing up the curl over a region is the line integral along the boundary.

One can push the discretisation further by assuming that the functions take values in a finite set. The integral theorems still work in that case too.

QUANTUM MULTIVARIABLE CALCULUS. Quantum calculus is "calculus without taking limits". There are indications that space and time look different at the microscopic small, the Planck scale of the order $\hbar$. One of the ideas to deal with this situation is to introduce quantum calculus which comes in different types. We discuss q-calculus where the derivative is defined as $\boxed{D_q f(x) = d_q f(x)/d_q(x)}$ with $\boxed{d_q f(x) = f(qx) - f(x)}$. You can see that $D_q x^n = [n] x^{n-1}$, where $[n] = \frac{q^n - 1}{q-1}$. As $q \to 1$ which corresponds to the deformation of quantum mechanics with $\hbar \to 0$ to classical mechanics, we have $[n] \to n$. There are quantum versions for differentiation rules like $D_q(fg)(x) = D_q f(x) g(x) + f(qx) D_q g(x)$ but quantum calculus is more friendly to students because **there is no simple chain rule**.

Once, we can differentiate, we can take anti-derivatives. It is denoted by $\int f(x)\, d_q(x)$. As we know the derivative of x^n, we have $\int x^n\, dq(x) = \sum_n a_n x^{n+1}/[n+1] + C$ with a constant C. The anti-derivative of a general function is a series $\boxed{\int f(x)\, d_q(x) = (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x)}$ For functions $f(x) \leq M/x^\alpha$ with $0 \leq \alpha < 1$, the integral is defined. With an anti-derivative, there are definite integrals. The **fundamental theorem of q-calculus** $\int_a^b f(x) d_q(x) = F(b) - F(a)$ where F is the anti-derivative of f still holds. Many results generalize to q-calculus like the Taylor theorem. A book of Kac and Cheung is a beautiful read about that.

Multivariable calculus and differential equations can be developed too. A handicap is the lack of a chain rule. For example, to define a line integral, we would have to define $\int F\, d_q r$ in such a way that $\int \nabla f\, dr_q = f(r(b)) - f(r(a))$. In quantum calculus, the naive definition of the length of a curve depends on the parameterization of the curve. Surprises with **quantum differential equations** QODE $d_q f = f$ which is $f(qx) - f(x) = (q-1)f(x)$ simplifies to $f(qx) = qf(x)$. It has solutions $f(x) = ax$, where a is a constant as well as functions obtained by taking an arbitrary function $g(t)$ on the interval $[1,q)$ satisfying $f(q) = qf(1)$ and extending it to the other intervals $[q^k, q^{k+1}]$ using the rule $f(q^k x) = q^k f(x)$. These solutions grow linearly. We have infinitely many solutions.

INFINITE DIMENSIONAL CALCULUS. Calculus in infinite dimensions is called **functional analysis**. Functions on infinite dimensional spaces which are also called **functionals** for which the gradient can be defined. The later is the analogue of $D_u f$. One does not always have $D_u f = \nabla f \cdot u$. An example of an infinite dimensional space is the set X of all continuous functions on the unit interval. An example of a two dimensional surface in that space would be $r(u,v) = (\cos(u)\sin(v)x^2 + \sin(u)\sin(v)\cos(x) + \cos(v)/(1+x^2)$. This surface is actually a two dimensional sphere. On this space X one can define a dot product $f \cdot g = \int f(x)g(x)\, dx$.

The theory which deals with the problem of extremizing functionals in infinite dimensions is called **calculus of variations**. There are problems in this field, which actually can be answered within the realm of multivariable calculus. For example: a classical problem is to find among all closed regions with boundary of length 1 the one which has maximal area. The solution is the circle. One of the proofs which was found more than 100 years ago uses Greens theorem. One can also look at the problem to find the polygon with n edges and length 1 which has maximal area. This is a Lagrange extremization problem with regular polygons as solutions. In the limit when the number of points go to infinity, one obtains the **isoperimetric inequality**. Other problems in the calculus of variations are the search for the shortest path between two points in a hilly region. This shortest path is called a **geodesic**. Also here, one can find approximate solutions by considering polygons and solving an extremization problem but direct methods in that theory are better.

GENERALIZED CALCULUS. In physics, one wants to deal with objects which are more general than functions. For example, the vector field $F(x,y) = (-y,x)/(x^2+y^2)$ has its curl concentrated on the origin $(0,0)$. This is an example of an object which is a **distribution**. An example of such a Schwartz distribution is a "function" f which is infinite at 0, zero everywhere else, but which has the property that $\int f d\ dx = 1$. It is called the **Dirac delta function**. Mathematically, one defines distributions as a linear map on a space of smooth "test functions" ϕ which decay fast at infinity. One writes (f,ϕ) for this. For continuous functions one has $(f,\phi) = \int_{-\infty}^{\infty} f(x)\phi(x)\ dx$, for the Dirac distribution one has $(f,\phi) = \phi(0)$. One would define the derivative of a distribution as $(f',\phi) = -f(\phi')$. For example for the Heavyside function $H(x) = \left\{ \begin{array}{l} 0\ x < 0 \\ 1\ x > 0 \end{array} \right.$ one has $(H,\phi) = \int_{-\infty}^{\infty} \phi(x)\ dx$ and because $(H,\phi') = \int_{-\infty}^{\infty} \phi'(x)\ dx = -\phi'(0)$ one has $(H',\phi) = \phi(0)$ the Dirac delta function. The Dirac delta function is still what one calls a **measure**, an object which appears also in probability theory. However, if we differentiate the Dirac delta function, we obtain $(D',\phi) = -(D,\phi') = -\phi'(0)$. This is an object which can no more be seen as a probability distribution and is a new truly "generalized" function.

COMPLEX CALCULUS Calculus in the complex is called **complex analysis**. Many things which are a bit mysterious in the real become more transparent when considered in the complex. For example, complex analysis helps to solve some integrals, it allows to solve some problems in the real plane better. Here are some places where complex analysis could have helped us: to find harmonic functions, one can take a nice function in the complex like $z^4 = (x+iy)^4 = x^4 - x^2y^2 + y^4 + i(x^3y + xy^3)$ and look at its real and complex part. These are harmonic functions. Differentiation in the complex is defined as in the real but since the complex plane is two dimensional, one asks more: $f(z+dz) - f(z)/dz$ has to exist and be equal for every $dz \to 0$. One writes $\partial_z = (\partial_x - i\partial_y)/2$ and **complex differentiation** satisfies all the known properties from differentiation on the real line. For example $d/dz z^n = nz^{n-1}$. Multivariable calculus very much helps also to integrate functions in the complex. Again, because the complex plane is two dimensional, one can integrate along paths and the **complex integral** $\int_C f(z)\ dz$ is actually a line integral. If $z(t) = x(t) + iy(t)$ is the path and $f = u + iv$ then $\boxed{\int_C f(z)\ dz = \int_a^b (ux' - vy')dt + i\int_a^b (uy' + vx')\ dt}$. Greens theorem for example is the easiest way to derive the Cauchy integral theorem which says that $\int_C f(z)\ dz = 0$ if C is the boundary of a region in which f is differentiable. Integration in the complex is useful for example to compute definite integrals.

One can ask, whether calculus can also be done in other number systems besides the reals and the complex numbers. The answer is yes, one can do calculus using **quaternions**, **octonions** or over finite fields but each of them has its own difficulty. Quaternion multiplication already does no more commute $AB \neq BA$ and octonions multiplication is even no more associative $(AB)C \neq A(BC)$. In order to do calculus in finite fields, differentiation will have to be replaced by differences. A Finnish mathematician Kustaanheimo promoted around 1950 a finite geometrical approach with the aim to do real physics using a very large prime number. These ideas were later mainly picked up by philosophers. It is actually a matter of fact that the **natural numbers** are the most complex and the **complex numbers** are the **most natural**.

CALCULUS IN HIGHER DIMENSIONS. When extending calculus to higher dimensions, the concept of vectorfields, functions and differentiations grad,curl,div are reorganized by introducing **differential forms**. For an integer $0 \leq k \leq n$, define **k-forms** as objects of the form $\boxed{\alpha = \sum_I a_I dx_I = \sum_{i_1<i_2<\ldots<i_k} a_{i_1,\ldots,i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}}$, where $dx_i \wedge dx_j$ is a multiplication called **exterior product** which is anti-commutative: $\alpha \wedge \beta = -\beta \wedge \alpha$. Then, one defines an **exterior derivative** as $d\sum_I a_I dx_I = da_I \wedge dx_I$ In three dimensions, where we write $dx = dx_1, dy = dx_2$ and $dz = dx_3$:

$k=$	form $\alpha =$	$d\alpha =$
0	f	$f_x dx + f_y dy + f_z dz$
1	$Mdx + Ndy + Pdz$	$(N_x - M_y)dx \wedge dy + (P_x - M_z)dx \wedge dz + (P_y - N_z)dy \wedge dz$
2	$Adx_1 \wedge dx_2 + Bdx_1 \wedge dx_3 + Cdx_2 \wedge dx_3$	$(A_z - B_y + C_x)dx \wedge dy \wedge dz$
3	$gdx_1 \wedge dx_2 \wedge dx_3$	0.

(The computation of the curl was $N_x dx \wedge dy + P_x dx \wedge dz + M_y dy \wedge dx + P_y dy \wedge dz + M_z dz \wedge dx + N_z dz \wedge dy = (N_x - M_y)dx \wedge dy + (P_x - M_z)dx \wedge dz + (P_y - N_z)dy \wedge dz$.) After an identification of 0 with 3 forms and 1 with 2 forms (called **Hodge *operation**) one can see the exterior derivatives as gradient, curl and divergence:

It is useful also to clarify the notion of a surface by defining **manifolds**. Many definitions which we have seen for curves or surfaces can be extended to manifolds. The dot product can be defined more generally on manifolds and leads to a theory called **Riemannian geometry** used in the theory of general relativity. When learning relativity, one deals with 4-dimensional manifolds which incorporate both space and time. There, the language of differential forms is already mandatory. The exterior derivative of a 1 form is a 2 form which has 6 component. An example of such a 2-form is the electromagnetic field $F = dA$ combining 3 electric and 3 magnetic components. The **Maxwell equations** are $dF = 0, d^*F = I$, where $d^* = *d*$ is defined using the Hodge * operation. The consequence $d^*dA = I$ becomes the wave equation $d^*dA = 0$ in the absence of I a vector incorporating both electric charge and current i.

FRACTAL CALCULUS. We have dealt in this courses with 0-dimensional objects (points), 1-dimensional objects (curves), 2-dimensional objects (surfaces) as well as 3-dimensional objects (solids). Since more than 100 years, mathematicians also studied **fractals**, objects with non-integer dimension are called fractals. An example is the Koch snowflake which is obtained as a limit by repeated stellation of an equilateral triangle of initial arc-length $A = 3$. After one stellation, the length has increased by a factor $4/3$. After n steps, the length of the curve is $A(4/3)^n$. The dimension is $\log(4/3) > 1$. There are things which do no more work here. For example, the length of this curve is infinite. Also the curve has no defined velocity at all places. One can ask whether one could apply Greens theorem still in this case. In some sense, this is possible. After every finite step of this construction one can compute the line integral along the curve and Greens theorem tells that this is a double integral of $\mathrm{curl}(F)$ over the region enclosed by the curve. Since for larger and large n, less and less region is added to the curve, the integral $\int\int \mathrm{curl}(F)\ dxdy$ is defined in the limit. So, one can define the line integral along the curve. Closely related to fractal theory is **geometric measure theory**, which is a generalization of differential geometry to surfaces which are no more smooth. Merging in ideas from generalized functions and differential forms, one defines **currents**, functionals on smooth differential forms. The theory is useful for studying minimal surfaces. Fractals appear naturally in differential equations as attractors. The most infamous fractal is probably the Mandelbrot set, which is defined as the set of complex numbers c for which the iteration of the map $f(z) = z^2 + c$ starting with 0 leads to a bounded sequence $0 \to c \to c^2 + c \to (c^2+c)^2 + c....$ The boundary of the Mandelbrot set is actually not a fractal. It is so wiggly that its dimension is actually 2. One can modify the Koch snowflake to get dimension 2 too. If one adds new triangles each time so that the length of the new curve is doubled each time, then the dimension of the Koch curve is 2 too.

THE FOUNDATIONS OF CALCULUS. Would it be possible that aliens somewhere else would build up mathematics radically different, by starting with a different axiom system? It is likely. The reason is that already we know that there is not a single way to build up mathematical truth. We have some choice: it came as a shock around the middle of the last century that for any strong enough mathematical theory, one can find statements which are **not provable** within that system and which one can either accept as a new axiom or accept the negation as a new axiom. Even some of the respected axioms are already independent of more elementary ones. One of them is the **axiom of choice** which says that for any collection C of nonempty set, one can choose from each set an element and form a new set. A consequence of this axiom which is close to calculus is a "compactness property" for functions on the interval. If we take the distance $d(f,g) = \max(f(x) - g(x))$ between two functions and take a sequence of differentiable functions satisfying $|f_n(x)| \leq M$ and $|f_n'(x)| \leq M$ then there exists a subsequence f_{n_k} which converges to a continuous function.

A rather counterintuitive consequence of the Axiom of choice is that one can decompose the unit ball into 5 pieces E_i, move those pieces using rotations and translations and reassemble them to form two copies of the unit ball. We can not integrate $\int\int\int_{E_i}\ dV$ over this regions: the sum of their volumes would be either $4\pi/3$ or $8\pi/3$ depending on whether we integrate before or after the arrangement. This is such a paradoxical construction that one calls it a **paradox**: the **Banach-Taski** paradox.

An other axiom for which is not clear, whether it matters in calculus is the continuum hypothesis which tells that there is no cardinality between the "infinity" of the natural numbers $\aleph_0$ and the "infinity" of the real numbers α_1.

It had been realized in the 19'th century by Cantor that these two infinities are different. Cantors argument is to assume that one could enumerate all numbers between 0 and 1: then look at the diagonal number, in which each digit is altered. For example: $\boxed{x = 0.35285....}$. This real number x disagrees with all the numbers in the list and was therefore not accounted for in the counting.

1 0.[2]42314234123412341 34...
2 0.3[4]42234982734139041 73...
3 0.69[1]43410741473467388 74...
4 0.999[7]42833824642001041 31...
5 0.3620[4]47473892389342111 47...

The possibilities to question or alter the mathematical build up does not stop on the set theoretical level. People even tried to change **logic**. Among things which have been proposed are **fuzzy logic**, where truth can take a value between 0 and 1 or quantum logic or a logic in which one has three possibilities, true, not true or "undecided".

One not only has the possibility to extend mathematics differently. A rather rich playground can be covered by **restricting tools**. One can for example ask, what part of calculus can still be done in without the axiom of choice. One can also ask that one only is allowed to talk about objects which can be constructed explicitly. There were people, **strict finitists**, who went even further and suggested to disqualify too large numbers like $10^{10^{10}}$. Calculus teachers have even to go further and produce problems so that the answer is 1 or 0 or π or 1/3. Problems in final exams, where the answer is 1234/12 is unthinkable. The philosophical direction teachers are forced to follow is called **ultra strict finitism**...