Lecture 32: Trig substitutions

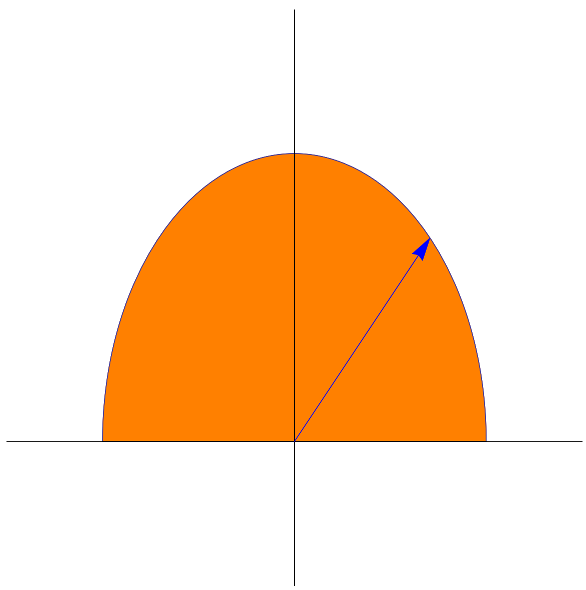
A **Trig substitution** is a special substitution, where x is a trigonometric function of u or u is a trigonometric function of x. Also this topic is covered more in follow up courses like Math 1b. This lecture allows us to practice more the substitution method. Here is an important example:

1 The area of a half circle of radius 1 is given by the integral

$$\int_{-1}^{1} \sqrt{1-x^2}\, dx \ .$$

Solution. Write $x = \sin(u)$ so that $\cos(u) = \sqrt{1-x^2}$. $dx = \cos(u)du$. We have $\sin(-\pi/2) = -1$ and $\sin(\pi/2) = 1$ the answer is

$$\int_{-\pi/2}^{\pi/2} \cos(u)\cos(u) du = \int_{-\pi/2}^{\pi/2} (1+\cos(2u))/2 = \frac{\pi}{2} \ .$$



Lets generalize this a bit and do the same computation for a general radius r:

2 Compute the area of a half disc of radius r which is given by the integral

$$\int_{-r}^{r} \sqrt{r^2-x^2}\, dx \ .$$

Solution. Write $x = r\sin(u)$ so that $r\cos(u) = \sqrt{r^2-x^2}$ and $dx = r\cos(u)\ du$ and $r\sin(-\pi/2) = -r$ and and $r\sin(\pi/2) = r$. The answer is

$$\int_{-\pi/2}^{\pi/2} r^2\cos^2(u)\ du = r^2\pi/2 \ .$$

Here is an example, we know how to integrate

3 Find the integral

$$\int \frac{dx}{\sqrt{1-x^2}} \ .$$

We know the answer is $\arcsin(x)$. How can we do that without knowing? **Solution.** We can do it also with a trig substitution. Try $x = \sin(u)$ to get $dx = \cos(u)\, du$ and so

$$\int \frac{\cos(u)\, du}{\cos(u)} = u = \arcsin(x) + C\ .$$

Here is an example, where $\tan(u)$ is the right substitution. You have to be told that first. It is hard to come up with the idea:

4 Find the following integral:

$$\int \frac{dx}{x^2\sqrt{1+x^2}}$$

by using the substitution $x = \tan(u)$. **Solution.** Then $1 + x^2 = 1/\cos^2(u)$ and $dx = du/\cos^2(u)$. We get

$$\int \frac{du}{\cos^2(u)\tan^2(u)(1/\cos(u))} = \int \frac{\cos(u)}{\sin^2(u)}\, du = -1/\sin(u) = -1/\sin(\arctan(x))\ .$$

Trig substitution is based on the trig identity :

$$\cos^2(u) + \sin^2(u) = 1$$

Depending on whether you divide this by $\sin^2(u)$ or $\cos^2(u)$ we get

$$1 + \tan^2(u) = 1/\cos^2(u),\ 1 + \cot^2(u) = 1/\sin^2(u)$$

These identities are worth remembering. Lets look at more examples:

5 Evaluate the following integral

$$\int x^2/\sqrt{1-x^2}\, dx\ .$$

Solution: Substitute $x = \cos(u), dx = -\sin(u)\, du$ and get

$$\int -\frac{\cos^2(u)}{\sin(u)}\sin(u)du = -\int \cos^2(u)\, du = -\frac{u}{2} - \frac{\sin(2u)}{4} + C = -\frac{\arcsin(x)}{2} + \frac{\sin(2\arcsin(x))}{4} + C\ .$$

6 Evaluate the integral

$$\int \frac{dx}{(1+x^2)^2}\ .$$

Solution: we make the substitution $x = \tan(u), dx = du/(\cos^2(u))$. Since $1 + x^2 = \cos^{-2}(u)$ we have

$$\int \frac{dx}{(1+x^2)^2} = \int \cos^2(u)\, du = (u/2) + \frac{\sin(2u)}{4} + C = \frac{\arctan(u)}{2} + \frac{\sin(2\arctan(u))}{4} + C\ .$$

Here comes an other prototype problem:

7 Find the anti derivative of $1/\sin(x)$. **Solution**: We use the substitution $u = \tan(x/2)$ which gives $x = 2\arctan(u), dx = 2du/(1+u^2)$. Because $1 + u^2 = 1/\cos^2(x/2)$ we have

$$\frac{2u}{1+u^2} = 2\tan(x/2)\cos^2(x/2) = 2\sin(x/2)\cos(x/2) = \sin(x)\ .$$

Plug this into the integral

$$\int \frac{1}{\sin(x)}\, dx = \int \frac{1+u^2}{2u}\frac{2du}{1+u^2} = \int \frac{1}{u}\, du = \log(u) + C = \log(\tan(\frac{x}{2})) + C\ .$$

Unlike before, where x is a trig function of u, now u is a trig function of x. This example shows that the substitution $u = \tan(x/2)$ is magic. Because of the following identities

0. $u = \tan(x/2)$
$\boxed{1}$ $dx = \frac{2du}{(1+u^2)}$
$\boxed{2}$ $\sin(x) = \frac{2u}{1+u^2}$
$\boxed{3}$ $\cos(x) = \frac{1-u^2}{1+u^2}$

It allows us to reduce any rational function involving trig functions to rational functions. [1]

Any function $p(x)/q(x)$ where p, q are trigonometric polynomials can be integrated using elementary functions.

It is usually a lot of work but here is an example:

8 To find the integral

$$\int \frac{\cos(x) + \tan(x)}{\sin(x) + \cot(x)} \, dx$$

for example, we replace $dx, \sin(x), \cos(x), \tan(x) = \sin(x)/\cos(x), \cot(x) = \cos(x)/\sin(x)$ with the above formulas we get a rational expression which involves u only This gives us an integral $\int p(u)/q(u) \, du$ with polynomials p, q. In our case, this would simplify to

$$\int \frac{2u\,(u^4 + 2u^3 - 2u^2 + 2u + 1)}{(u-1)(u+1)\,(u^2+1)\,(u^4 - 4u^2 - 1)} du$$

The method of partial fractions provides us then with the solution.

[1] Proofs: $\boxed{1}$ differentiate to get $du = dx/(2\cos^2(x/2)) = dx(1+u^2)/2$. $\boxed{2}$ use double angle $\sin(x) = 2\tan(x/2)\cos^2(x/2)$ and then $1/\cos^2(x/2) = 1 + \tan^2(x/2)$. $\boxed{3}$ use double angle $\cos(x) = \cos^2(x/2) - \sin^2(x/2) = (1 - \sin^2(x/2)/\cos^2(x/2))\cos^2(x/2)$ and again $1/\cos^2(x/2) = 1 + \tan^2(x/2)$.

Homework

1 Find the anti-derivative:

$$\int \sqrt{1-9x^2}\; dx\ .$$

2 Find the anti-derivative:

$$\int (1-x^2)^{3/2}\; dx\ .$$

3 Find the anti-derivative:

$$\int \frac{\sqrt{1-x^2}}{x^2}\; dx\ .$$

4 Integrate

$$\int \frac{dx}{1+\sin(x)}\ .$$

Use the substitution $u = \tan(x/2)$.

5 Compute

$$\int_0^{\pi/3} \frac{dx}{\cos(x)}$$

using the substitution $u = \tan(x/2)$. Instead of backsubstitution, you can also substitute the bounds.