Lecture 19: Fundamental theorem

In this lecture we come back to the **fundamental theorem of calculus** for differentiable functions. This will allow us in general to compute integrals of functions which appear as derivatives. You have already made use of this theorem when doing the homework for today. We have seen earlier that with $Sf(x) = h(f(0) + \cdots + f(kh))$ and $Df(x) = (f(x+h) - f(x))/h$ we have $SDf = f(x) - f(0)$ and $DSf(x) = f(x)$ if $x = nh$. This becomes now:

Fundamental theorem of calculus: Assume f is differentiable and f' is continuous. Then

$$\int_0^x f'(t)\, dt = f(x) - f(0) \text{ and } \frac{d}{dx}\int_0^x f(t)\, dt = f(x)$$

Proof. Using notation of Euler, we write $A \sim B$. We say "A and B are close" and mean $A - B \to 0$ for $h \to 0$. From $DSf(x) = f(x)$ for $x = kh$ we have $DSf(x) \sim f(x)$ for $kh < x < (k+1)h$ because f is continuous. We also know $\int_0^x Df(t)\, dt \sim \int_0^x f'(t)\, dt$ because $Df(t) \sim f'(t)$ uniformly for all $0 \le t \le x$ by the definition of the derivative and the assumption that f' is continuous and using Bolzano on the bounded interval. We also know $SDf(x) = f(x) - f(0)$ for $x = kh$. By definition of the Riemann integral, $Sf(x) \sim \int_0^x f(t)\, dt$ and so $SDf(x) \sim \int_0^x Df(t)\, dt$.

$$f(x) - f(0) \sim SDf(x) \sim \int_0^x Df(t)\, dt \sim \int_0^x f'(t)\, dt$$

as well as

$$f(x) \sim DSf(x) \sim D\int_0^x f(t)\, dt \sim \frac{d}{dx}\int_0^x f(t)\, dt\ .$$

Bolzano and Weierstrass would write $A \sim B$ using $\epsilon - \delta$ notation: $\forall \epsilon > 0 \exists \delta > 0$ so that $|h| < \delta$ implies $|A - B| < \epsilon$. The $\epsilon - \delta$ notation is still used today but not intuitive. We avoid it thus.

1 $\int_0^5 3x^7\, dx = \frac{x^8}{8}|_0^5 = \frac{5^8}{8}$. You can always leave such expressions as your final result. It is even more elegant than the actual number 390625/8.

2 $\int_0^{\pi/2} \cos(x)\, dx = \sin(x)|_0^{\pi/2} = 1$.

3 In the following example, the result is in terms of a variable x. To integrate we have to use an other variable: $\int_0^x \sqrt{1+t}\, dt = \int_0^x (1+t)^{1/2}\, dt = (1+t)^{3/2}(2/3)|_0^x = [(1+x)^{3/2} - 1](2/3)$. You have seen that $1+t$ in the interior of the function does not make a big difference. Keep such examples in mind.

4 Also in this example $\int_0^2 \cos(t+1)\, dt = \sin(x+1)|_0^2 = \sin(2) - \sin(1)$, the additional term $+1$ does not make a big dent.

5 $\int_{\pi/6}^{\pi/4} \cot(x)\, dx$. This is an example where the anti derivative is difficult to spot. It is easy if we know where to look for: the function $\log(\sin(x))$ has the derivative $\cos(x)/\sin(x)$. So, we know the answer is $\log(\sin(x))|_{\pi/6}^{\pi/4} = \log(\sin(\pi/4)) - \log(\sin(\pi/6)) = \log(1/\sqrt{2}) - \log(1/2) = -\log(2)/2 + \log(2) = \log(2)/2$.

6 The example $\int_1^2 1/(t^2-9)\, dt$ is a bit challenging. We need a hint and write $-6/(x^2-9) = 1/(x+3) - 1/(x-3)$. The function $f(x) = \log|x+3| - \log|x-3|$ has therefore $-6/(x^2-9)$ as a derivative. We know therefore $\int_1^2 -6/(t^2-9)\, dt = \log|3+x| - \log|3-x||_1^2 = \log(5) - \log(1) - \log(4) + \log(2) = \log(5/2)$. The original task is now $(-1/6)\log(5/2)$.

7 $\int_0^x \cos(\sin(x))\cos(x)\, dx = \sin(\sin(x))$ because the derivative of $\sin(\sin(x))$ is $\cos(\sin(x))\cos(x)$. The function $\sin(\sin(x))$ is called the **antiderivative** of f. If we differentiate this function, we get $\cos(\sin(x))\cos(x)$.

8 Find $\int_0^\pi \sin(x)\, dx$. **Solution**: This has a very nice answer.

Here is an important notation, which we have used in the example and which might at first look silly. But it is a handy intermediate step when doing the computation.

$F|_a^b = F(b) - F(a)$.

We give reformulations of the fundamental theorem in ways in which it is mostly used:

If f is the derivative of a function F then

$$\int_a^b f(x)\, dx = F(x)|_a^b = F(b) - F(a)\ .$$

In some textbooks, this is called the "second fundamental theorem" or the "evaluation part" of the fundamental theorem of calculus. The statement $\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$ is the "antiderivative part" of the fundamental theorem. They obviously belong together and are two different sides of the same coin.

Here is a version of the fundamental theorem, where the boundaries are functions of x.

Given functions g, h and if F is a function such that $F' = f$, then

$$\int_{h(x)}^{g(x)} f(t)\, dt = F(g(x)) - F(h(x))\ .$$

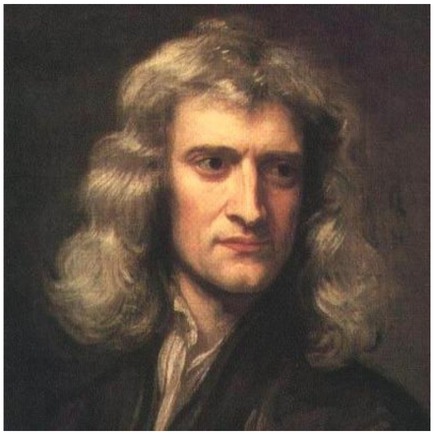
9 $\int_{x^4}^{x^2} \cos(t)\, dt = \sin(x^2) - \sin(x^4)$.

The function F is called an antiderivative. It is not unique but the above formula does always give the right result.

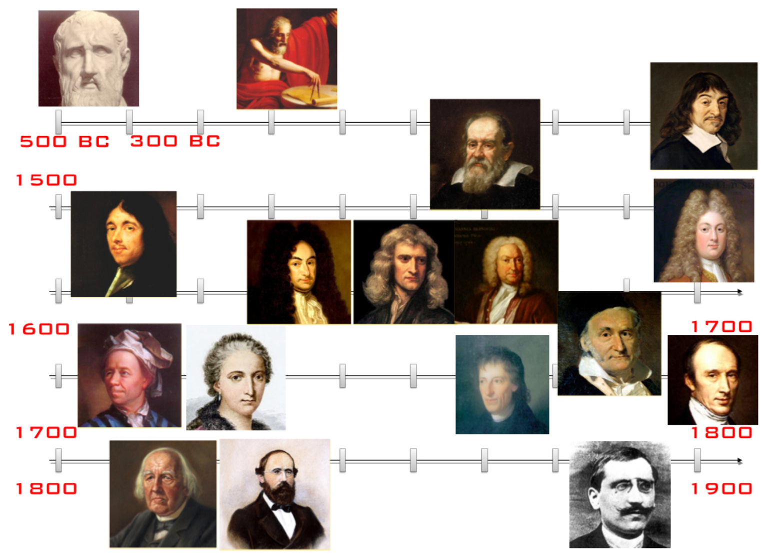
Lets look at a list of important antiderivatives. You should have as many antideriatives "hard wired" in your brain. It really helps. Here are the core functions you should know. They appear a lot.

function	anti derivative
x^n	$\frac{x^{n+1}}{n+1}$
$\sqrt{x}$	$\frac{x^{3/2}2}{3}$
e^{ax}	$\frac{e^{ax}}{a}$
$\cos(ax)$	$\frac{\sin(ax)}{a}$
$\sin(ax)$	$\frac{-\cos(ax)}{a}$
$\frac{1}{x}$	$\log(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\log(x)$	$x\log(x) - x$

Make your own table!




Meet **Isaac Newton** and **Gottfried Leibniz**. They have discovered the fundamental theorem of calculus. You can see from the expression of their faces, they are not pleased that Oliver has added other calculus pioneers. The sour faces might also have to do with the fact that they have to live forever the same handout with **Austin Powers** and **Doctor Evil**! But thats ok. Celebrities deserve to suffer.



Zeno of Elea 490-430 notion of limit
Democritus 460-370 cone and pyramid
Eudoxus 408-355 BC method of exhaustion
Archimedes 287-212 BC circle and sphere
Johannes Kepler 1571-1630, velocity
Rene Descartes 1596-1650, tangents
Bonaventura Cavalieri 1598-1647
Pierre de Fermat 1601-1665 maxima
John Wallis 1616-1703 infinite series
Christiaan Huygens 1629-1695 waves
Blaise Pascal 1623-1662, triangle
Isaac Barrow 1630-1677 tangents
James Gregory 1638-1675 fund. thm.
Robert Hooke 1635-1703 square law

Isaac Newton 1643-1727 Fluxions
Gottfried Leibniz 1646-1716 notation
Michel Rolle 1652-1719 Rolles theorem
Guillaume de L'Hopital 1661-1704 law
Johann Bernoulli 1667-1748 textbook
Brook Taylor 1685-1731 series, difference
Leonard Euler 1707-1783 Basel problem
Maria Agnesi 1718-1799 textbook
Bernard Bolzano 1781-1848 $\epsilon - \delta$, extrema
Augustin Cauchy 1789-1857, continuity
Karl Weierstrass 1815-1897 foundation
Bernhard Riemann 1826-1866 integral
Henri Lebesgue 1875-1941 integration

Homework

1 For the following integrals $\int f$, find a function F such that $F' = f$, then integrate
a) $\int_0^2 4x^3 + 10x \, dx$.
b) $\int_0^1 6(x+4)^3 \, dx$.
c) $\int_2^3 5/x + 7/(x-1) \, dx$.
d) $\int_0^{\sqrt{\pi}} \cos(x^2)x + \sin(x^2)x \, dx$

2 Find the following integrals by finding a function F satisfying $F' = f$. We will learn techniques to find the function. Here, we just use our knowledge about derivatives:
a) $\int_2^3 5x^4 + 4x^3 \, dx$.
b) $\int_{\pi/4}^{\pi/2} \sin(3x) + \cos(x) \, dx$.
c) $\int_{\pi/4}^{\pi/2} \frac{1}{\sin^2(x)} \, dx$.
d) $\int_2^3 \frac{1}{x-1} \, dx$.

3 Evaluate the following integrals:
a) $\int_1^2 2^x \, dx$.
b) $\int_{-1}^1 \cosh(x) \, dx$. (Remember $\cosh(x) = (e^x + e^{-x})/2$.)
c) $\int_0^1 \frac{1}{1+x^2} \, dx$.
d) $\int_{1/3}^{2/3} \frac{1}{\sqrt{1-x^2}} \, dx$.

4 a) Compute $F(x) = \int_0^{x^3} \sin(t) \, dt$, then find $F'(x)$.
b) Compute $G(x) = \int_{\sin(x)}^{\cos(x)} \exp(t) \, dt$ then find $G'(x)$

5 a) **A clever integral:** Evaluate the following integral:
$\int_0^{2\pi} \sin(\sin(\sin(\sin(\sin(x))))) \, dx$
Explain the answer you get.

b) **An evil integral**: Evaluate $\int_e^{e^e} \frac{1}{\log(x)x} \, dx$.
Hint: Can you figure out a function $f(x)$ which has $1/(\log(x)x)$ as the derivative?