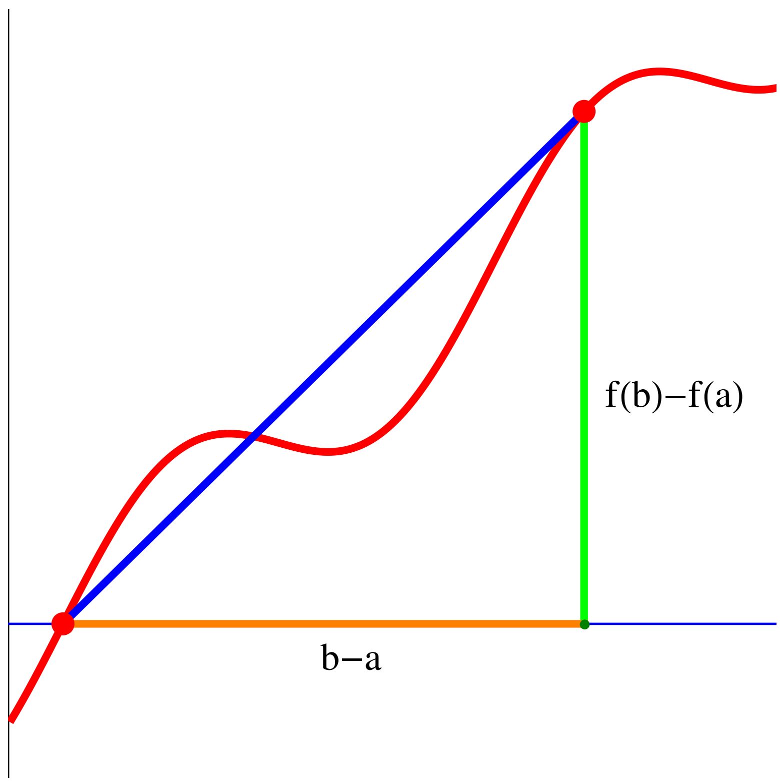
Lecture 16: The mean value theorem

In this lecture, we look at the **mean value theorem** and a special case called **Rolle's theorem**. Unlike the intermediate value theorem which applied for continuous functions, the mean value theorem involves derivatives. We assume therefore today that all functions are differentiable unless specified.

Mean value theorem: Any interval (a,b) contains a point x such that

$$f'(x) = \frac{f(b)-f(a)}{b-a} .$$



Here are a few examples which illustrate the theorem:

1 Verify with the mean value theorem that the function $f(x) = x^2 + 4\sin(\pi x) + 5$ has a point where the derivative is 1.
Solution. Since $f(0) = 5$ and $f(1) = 6$ we see that $(f(1) - f(0))/(1 - 0) = 5$.

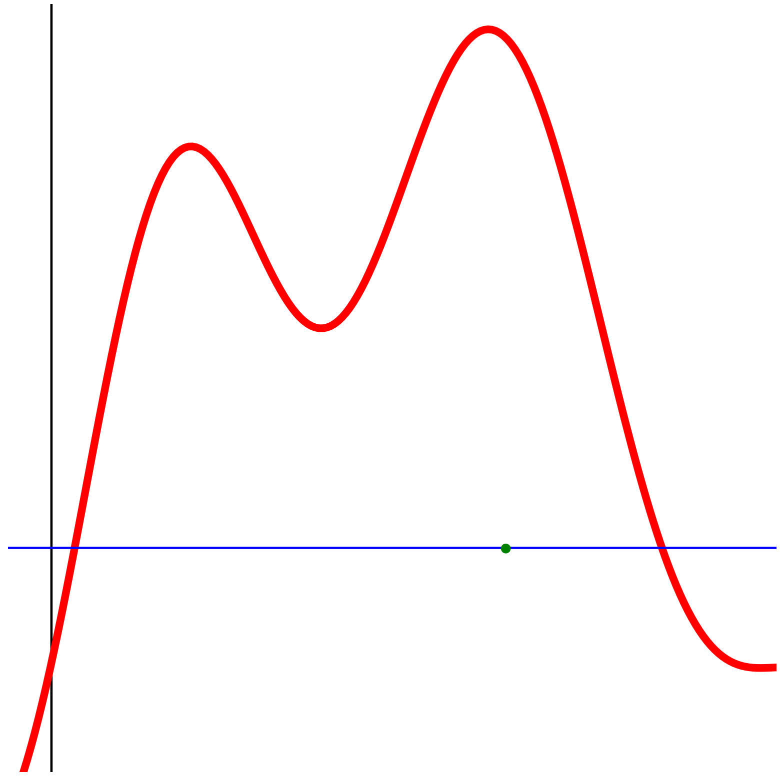
2 Verify that the function $f(x) = 4\arctan(x)/\pi - \cos(\pi x)$ has a point where the derivative is 3.
Solution. We have $f(0) = -1$ and $f(1) = 2$. Apply the mean value theorem.

3 A biker drives with velocity $f'(t)$ at position $f(b)$ at time b and at position a at time a. The value $f(b) - f(a)$ is the distance traveled. The fraction $[f(b) - f(a)]/(b - a)$ is the average speed. The theorem tells that there was a time when the bike had exactly the average speed.

4 The function $f(x) = \sqrt{1-x^2}$ has a graph on $(-1, 1)$ on which every possible slope is taken.
Solution: We can see this with the intermediate value theorem because $f'(x) = x/\sqrt{1-x^2}$ gets arbitrary large near $x = -1$ or $x = 1$. The mean value theorem shows this too because we can take intervals $[a, b] = [-1, -1 + c]$ for which $[f(b) - f(a)]/(b - a) = f(-1 + c)/ci \sim \sqrt{c}/c = 1/\sqrt{c}$ gets arbitrary large.

Proof of the theorem: the function $h(x) = f(a) + cx$, where $c = (f(b) - f(a)/(b - a)$ also connects the beginning and end point. The function $g(x) = f(x) - h(x)$ has now the property that $g(a) = g(b)$. If we can show that for such a function, there exists x with $g'(x) = 0$, then we are done. By tilting the picture, we have reduced the statement to a special case which is important by itself:

Rolle's theorem: If $f(a) = f(b)$ then f has a critical point in (a, b).



Proof: If it were not true, then either $f'(x) > 0$ everywhere implying $f(b) > f(a)$ or $f'(x) < 0$ implying $f(b) < f(a)$.

Second proof: Fermat's theorem assures a local maximum or local minimum of f exists in (a, b). At this point $f'(x) = 0$.

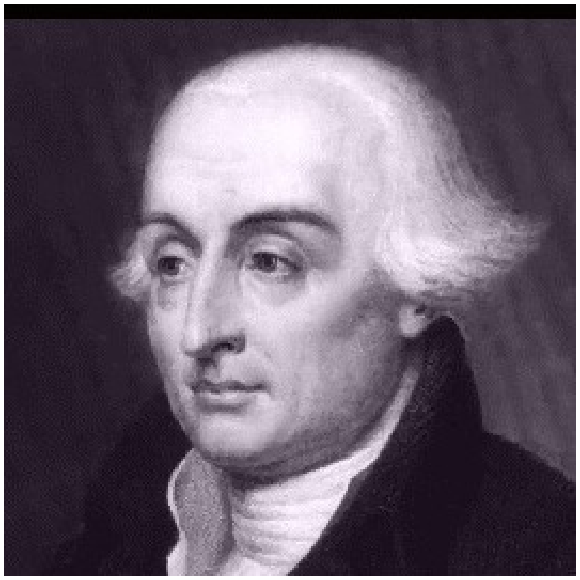
Recall that if f is continuous and $f(a) = f(b)$ then there is a local maximum or local minimum in the interval (a, b). This applies to every continuous function like $f(x) = |x|$ on $[-1, 1]$ which has a minimum at 0 without that $f'(x)$ exists at 0.

5 There is a point in $[0, 1]$ where $f'(x) = 0$ with $f(x) = x(1 - x^2)(1 - \sin(\pi x))$. **Solution:** We have $f(0) = f(1) = 0$. Use Rolle's theorem.

6 Show that the function $f(x) = \sin(x) + x(\pi - x)$ has a critical point $[0, \pi]$. **Solution:** The function is differentiable and nonnegative. It is zero at $0, \pi$. By Rolle's theorem, there is a critical point. Remark. We can not use Rolle's theorem to show that there is a local maximum even so the extremal value theorem assures us that this exist.

7 Verify that the function $f(x) = 2x^3 + 3x^2 + 6x + 1$ has only one real root. **Solution:** There is at least one real root by the intermediate value theorem: $f(-1) = -4, f(1) = 12$. Assume there would be two roots. Then by Rolle's theorem there would be a value x where $g(x) = f'(x) = 6x^2 + 6x + 6 = 0$. But there is no root of g. [The graph of g minimum at $g'(x) = 6 + 12x = 0$ which is 1/2 where $g(1/2) = 21/2 > 0$.]

It is not clear who discovered the **mean value theorem**? **Joseph Louis Lagrange** is a candidate. Also **Augustin Louis Cauchy** is credited for a modern formulation of the theorem.



Joseph Louis Lagrange, 1736-1813.

Augustin Louis Cauchy, 1789-1857.

What about **Michel Rolle**? He lived from 1652 to 1719, mostly in Paris. No picture of him seems available. Rolle also introduced the n'th root notation like when writing the cube root as

$$\sqrt[3]{x} \ .$$

It is still used today even so we prefer to write $x^{1/3}$ to make algebra easier. The identity $\sqrt[3]{x}\sqrt[3]{x^2} = x$ for example can be seen easier without the root symbol with $x^{1/3}x^{2/3} = x^1$.

Homework

1 Let $f(x) = \sin(5x) + \sin(x)$.
a) Use Rolle's theorem to see that f has a critical point on the interval $[0, 2\pi]$.
b) Use the mean value theorem to see that f has a point on $[\pi/2, 3\pi/2]$, where $f'(x) = -4/\pi = 1.2734....$
c) Use the mean value theorem to see that f has a point on $[5\pi/6, 7\pi/6]$, where $f'(x) = -6/\pi = 1.9098...$

2 a) The function $f(x) = 1 - |x|$ satisfies $f(-1) = f(1) = 0$ but there is no point in $(-1, 1)$ where $f'(x) = 0$. Why is this not a counter example to Rolle's theorem?
b) The function $f(x) = 1/x^2$ satisfies $f(-1) = f(1) = 1$ but there is no point in $(-1, 1)$ where $f'(x) = 0$. Why is this not a counter example to Rolle's theorem?

3 We look at the function $f(x) = x^{10} + x^4 - 20x$ on the positive real line. Use the **mean value theorem** on some interval (a, b) to assure the there exists x, where $f'(x) = 500$.

4 Write down the mean value theorem, the intermediate value theorem, the extreme value theorem and the Fermat theorem. Enter in the following table "yes" or "no", if the property is needed.

Property needed?	Mean value	Intermediate value	Extreme value	Fermat
f is continuous				
f is differentiable				
f is positive				

5 Use the mean value theorem to verify the following statement: if a function f has a minimum at a and a maximum at b, then there exists an inflection point between a and b.