Lecture 13: Hopitals rule

The rule

The Hopital's rule is a miracle procedure which solves all our worries about limits:

Hopital's rule. If f, g are differentiable and $f(p) = g(p) = 0$ and $g'(p) \neq 0$, then

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f'(x)}{g'(x)} .$$

Lets see how it works:

1 Lets prove **the fundamental theorem of trigonometry** again:

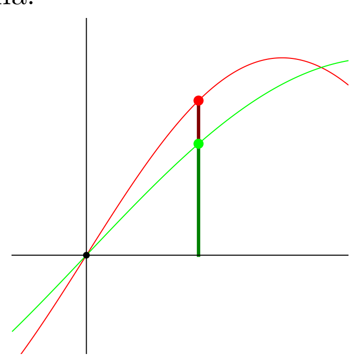
$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1 .$$

Why did we work so hard for this? We used the fundamental theorem to derive the derivatives for cos and sin at all points. In order to apply l'Hopital, we had to know the derivative. Our work to establish the limit was not in vain.

The proof of the rule is comic in its simplicity. Especially after we will see how fantastically useful it is:
since $f(p) = g(p) = 0$ we have $Df(p) = f(p+h)/h$ and $Dg(p) = g(p+h)/h$ so that for every $h > 0$ with $g(p+h) \neq 0$ the **quantum l'Hopital rule** holds:

$$\frac{f(p+h)}{g(p+h)} = \frac{Df(p)}{Dg(p)} .$$

Now take the limit $h \to 0$. Voilà!



Sometimes, we have to administer a medicine twice. To use this, l'Hopital can be improved in that the condition $g'(0) = 0$ can be replaced by the requirement that the limit $\lim_{x \to p} f'(x)/g'(x)$ exists. Instead of having a rule which replaces a limit with an other limit and cure a disease with a new one, we formulate it how it is used. The second derivative case could easily be generalized for higher derivatives. There is no need to memorize this. Just remember that you can check in several times to a hospital.

If $f(p) = g(p) = f'(p) = g'(p) = 0$ then $\lim_{x \to p} \frac{f(x)}{g(x)} = \lim_{x \to p} \frac{f''(x)}{g''(x)}$ if the limit to the right exists.

2 Find the limit $\lim_{x\to 0}(1-\cos(x))/x^2$. This limit had been pivotal to compute the derivatives of trigonometric functions. **Solution**: differentiation gives

$$\lim_{x\to 0} -\sin(x)/2x \ .$$

Now apply l'Hopital again.

$$\lim_{x\to 0} -\sin(x)/(2x) = \lim_{x\to 0} -\cos(x)/2 = -\frac{1}{2} \ .$$

3 **Problem.** Find the limit $f(x) = (\exp(x^2) - 1)/\sin(x^2)$ for $x \to 0$.

4 **Problem:** What do you get if you apply l'Hopital to the limit $[f(x+h) - f(x)]/h$ as $h \to 0$?
Answer: Differentiate both sides with respect to h! And then feel awsome!

5 Find $\lim_{x\to\infty} x\sin(1/x)$. **Solution.** Write $y = 1/x$ then $\sin(y)/y$. Now we have a limit, where the denominator and nominator both go to zero.

The case when both sides converge to infinity can be reduced to the 0/0 case by looking at $A = f/g = (1/g(x))/(1/f(x))$ which has the limit $g'(x)/g^2(x)/f'(x)/f^2(x) = g'(x)/f'(x)((1/g)/(1/f))^2 = g'/f'(f^2/g^2) = (g'/f')A^2$, so that $A = f'(p)/g'(p)$. We see:

If $\lim_{x\to p} f(x) = \lim_{x\to p} g(x) = \infty$ for $x \to p$ and $g'(p) \neq 0$, then

$$\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{f'(p)}{g'(p)} \ .$$

2 What is the limit $\lim_{x\to 0} x^x$? This answers the question **What is 0^0?**
Solution: Because $x^x = e^{x\log(x)}$, it is enough to understand the limit $x\log(x)$ for $x \to 0$.

$$\lim_{x\to 0} \frac{\log(x)}{1/x} \ .$$

Now the limit can be seen as the limit $(1/x)/(-1/x^2) = -x$ which goes to 0. Therefore $\lim_{x\to 0} x^x = 1$. (We assume that $x > 0$ in order to have real values x^x)

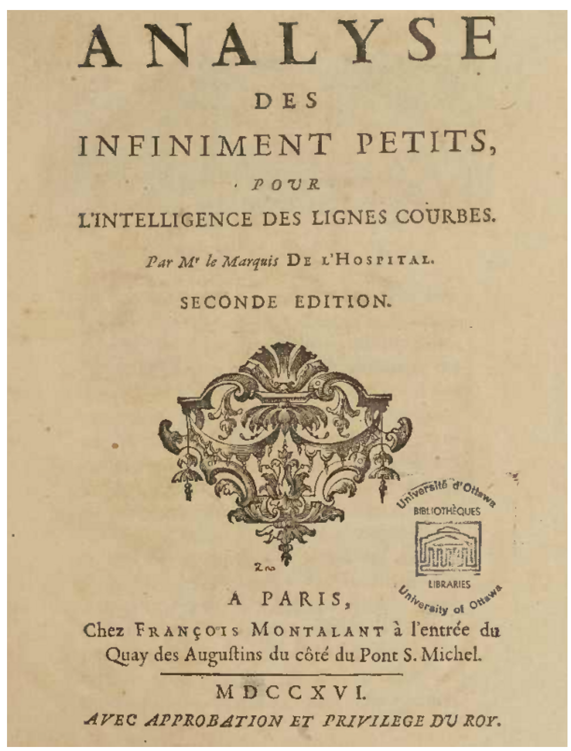
3 Find the limit $\lim_{x\to 2} \frac{x^2-4x+4}{\sin^2(x-2)}$.
Solution: this is a case where $f(2) = f'(2) = g(2) = g'(2) = 0$ but $g''(0) = 2$. The limit is $f''(2)/g''(2) = 2/2 = 1$.
Hopital's rule always works in calculus situations, where functions are differentiable. The rule can fail if differentiability of f or g fails. Here is a "rare" example:

4 **Deja Vue**: Find $\frac{\sqrt{x^2+1}}{x}$ for $x \to \infty$. L'Hopital gives $x/\sqrt{x^2+1}$ which in terms gives again $\frac{\sqrt{x^2+1}}{x}$. Apply l'Hopital again to get the original function. We got an infinite loop. If the limit is A, then the procedure tells that it is equal to $1/A$. The limit must therefore be 1. This case can be covered easily without l'Hopital: divide both sides by x to get $\sqrt{1+1/x^2}$. Now, we can see the limit 1.

5 **Trouble?** The limit $\lim_{x\to\infty}(2x+\sin(x))/3x$ is clearly 2/3 since we can take the sum apart. Hopital gives $\lim_{x\to\infty}(2+\cos(x))/3$ which has no limit. This is not trouble since Hopital applies only if the limit to right exists.

History

The "first calculus book", the world has known was "Analyse des Infiniment Petits pour l"intelligence des Lignes Courbes". It appeared in 1696 and was written by **Guillaume de l'Hopital**, a text if typeset in a modern font would probably fit onto 50-100 pages.[1] It is now clear that the mathematical content of Hopital's book is mostly due to **Johannes Bernoulli**: *Clifford Truesdell write in his article "The New Bernoulli Edition",* [2] *about this "most extraordinary agreement in the history of science": l'Hopital wrote: "I will be happy to give you a retainer of 300 pounds, beginning with the first of January of this year ... I promise shortly to increase this retainer, which I know is very modest, as soon as my affairs are somewhat straightened out ... I am not so unreasonable as to demand in return all of your time, but I will ask you to give me at intervals some hours of your time to work on what I request and also to communicate to me your discoveries, at the same time asking you not to disclose any of them to others. I ask you even not to send here to Mr. Varignon or to others any copies of the writings you have left with me; if they are published, I will not be at all pleased. Answer me regarding all this ..."* Bernoulli's response is lost, but a letter from l'Hopital indicates that it was quickly accepted. Clifford Truesdell also mentions that the book of l'Hopital has remained the standard for Calculus for a century.


ANALYSE
DES
INFINIMENT PETITS,
POUR
L'INTELLIGENCE DES LIGNES COURBES.
Par Mr le Marquis DE L'HOSPITAL.
SECONDE EDITION.
A PARIS,
Chez FRANÇOIS MONTALANT à l'entrée du Quay des Auguſtins du côté du Pont S. Michel.
MDCCXVI.
AVEC APPROBATION ET PRIVILEGE DU ROY.

[1] Stewart's book with 1200 pages probably contains about 4 million characters, about 12 times more than l'Hopital's book. It also contains more material of course. The OCR text of l'Hopital's book of 200 pages has 300'000 characters.

[2] Isis, Vol. 49, No. 1, 1958, pages 54-62

Homework

1 For the following functions, find the limits as $x \to 0$:
a) $8x/\sin(x)$
b) $(\exp(x) - 1)/(\exp(3x) - 1)$
c) $\sin^2(3x)/\sin^2(5x)$
d) $\frac{\sin(x^2)}{\sin^2(x)}$
e) $\sin(\sin(x))/x$.

2 For the following functions, find the limits as $x \to 1$:
a) $(x^2 - x - 1)/(\cos(x - 1) - 1)$
b) $(\exp(x) - e)/(\exp(3x) - e^3)$
c) $(x - 4)/(4x + \sin(x) + 8)$
d) Find the limit as $x \to \infty$:
$(x^2 - x - 1)/\sqrt{x^4 + 1}$.
(**Hint.** Find the limit of $(x^2 - x - 1)^2/(x^4 + 1)$ first, then take the square root of the limit).

3 Use l'Hopital to compute the following limits at $x = 0$:
a) $\log(5x)/\log|x|$.
b) $\lim_{x\to 0} 1/\log|x|$
c) $\text{sinc}'(x) = (\cos(x)x - \sin(x))/x^2$
d) $\log|\log|1 + x||/\log|\log|2 + x||$.

4 We have seen how to compute limits with healing. Solve the following healing problems with l'Hopital at $x = 1$:
a) $\frac{x^{1000}-1}{x^{20}-1}$.
b) $\frac{\tan^2(x-1)}{(\cos(x-1)-1)}$

5 More practice.
a) Find the limit $\lim_{x\to 0} \frac{1-e^x}{x-x^3}$.
b) Find the limit $\lim_{x\to 0} \frac{\log(1+3x)}{x}$.
c) Find the limit $\lim_{x\to 1}(x^5 - 1)/(x^3 - 1)$.
d) Find the limit $\lim_{x\to 0} \frac{4x}{\tan(5x)}$.