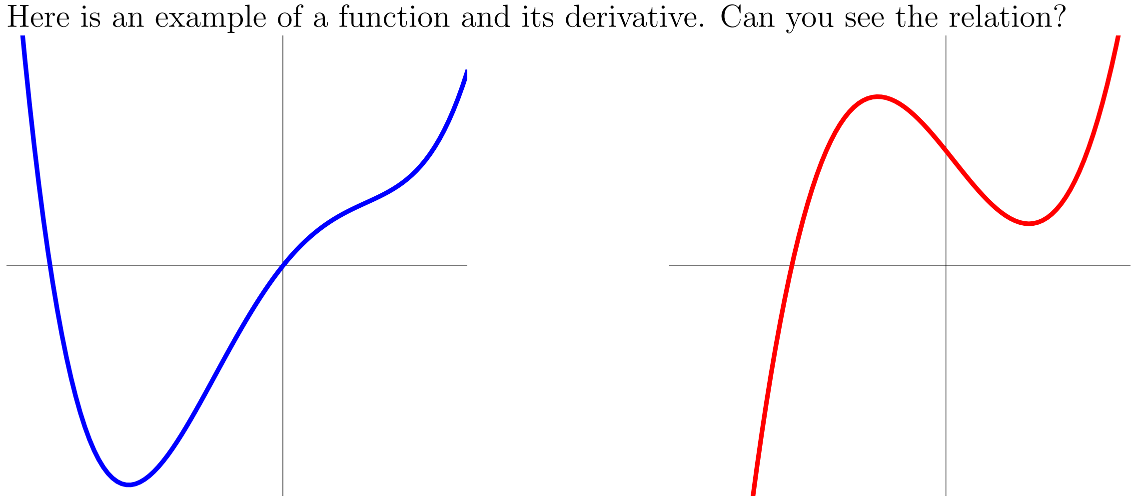
# Lecture 8: The derivative function

We have defined the derivative $f'(x) = \frac{d}{dx}f(x)$ as a limit of $(f(x+h) - f(x))/h$ for $h \to 0$. We have seen that $\frac{d}{dx}x^n = nx^{n-1}$ holds for integer $n$. We also know already that $\sin'(ax) = a\cos(x), \cos'(ax) = -a\sin(x)$ and $\exp'(ax) = a\exp(x)$. We can already differentiate a lot of functions and evaluate the derivative $f'(x)$ at some point $x$. This is the slope of the curve at $x$.

1 Find the derivative $f'(x)$ of $f(x) = \sin(4x) + \cos(5x) - \sqrt{x} + 1/x + x^4$ and evaluate it at $x = 1$. **Solution**: $f'(x) = 4\cos(4x) - 5\sin(5x) - 1/(2\sqrt{x} - 1/x^2 + 4x^3$. Plugging in $x = 1$ gives $-\pi - 1/2 - 1 + 4$.

The function which takes the derivative at a given point is called the **derivative function**. For example, for $f(x) = \sin(x)$, we get $f'(x) = \cos(x)$. In this lecture, we want to understand the new function and its relation with $f$. What does it mean if $f'(x) > 0$. What does it mean that $f'(x) < 0$. Do the roots of $f$ tell something about $f'$ or do the roots of $f'$ tell something about $f$?

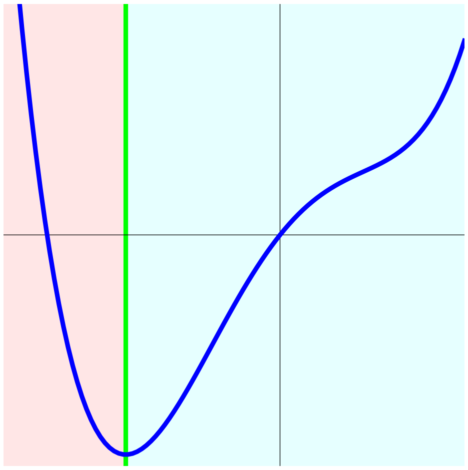
Here is an example of a function and its derivative. Can you see the relation?

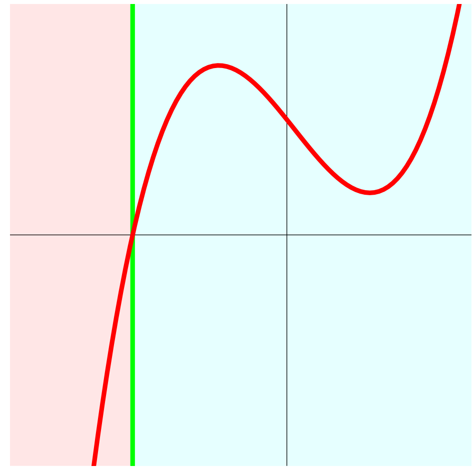


To understand the relation, it is good to distinguish intervals, where $f(x)$ is increasing or decreasing. This are the intervals where $f'(x)$ is positive or negative.

A function is called **monotonically increasing** on an interval $I = (a, b)$ if $f'(x) > 0$ for all $x \in (a, b)$. It is **monotonically decreasing** if $f'(x) < 0$ for all $x \in (a, b)$.

Monotonoically increasing functions “go up” when you “increase x”. Lets color that:





**2** Can you find a function $f$ on the interval $[0,1]$ which is bounded $|f(x)| \leq 1$ but such that $f'(x)$ is unbounded? Hint: square-"..."-beer.

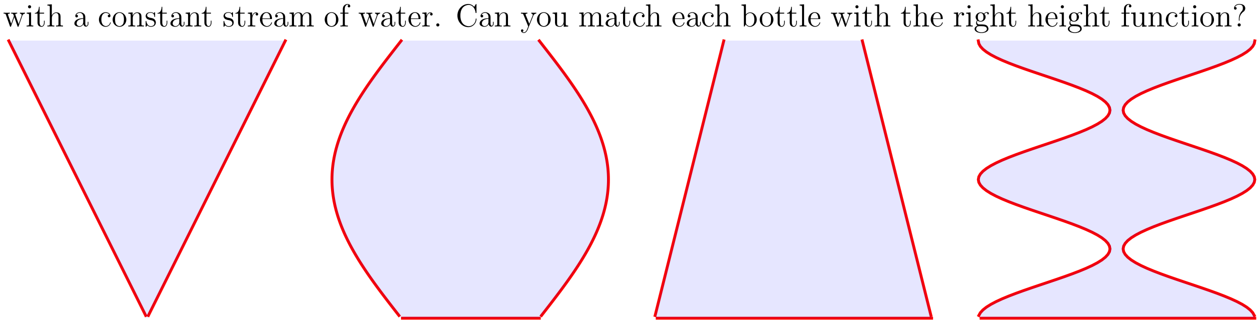
> Given the function $f(x)$, we can define $g(x) = f'(x)$ and then take the derivative $g'$ of $g$. This second derivative $f''(x)$ is called the **acceleration**. It measures the rate of change of the tangent slope. For $f(x) = x^4$, for example we have $f''(x) = 12x^2$. If $f''(x) > 0$ on some interval the function is called **concave up**, if $f''(x) < 0$, it is **concave down**.

**3** Find a function $f$ which has the property that its acceleration is constant equal to 6. **Solution.** We have to get a function such that its derivative is $6x$. That works for $3x^2$.

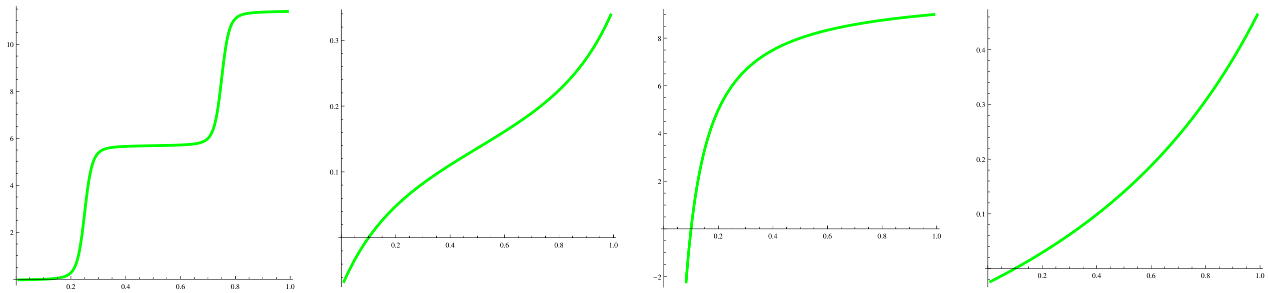
**4** Find a function $f$ which has the property that its acceleration is equal to its derivative. To do so, try basic functions you know and compute $f'(x), f''(x)$ in each case.

After matching some functions, we look at related inverse problem called **bottle calibration problem**. We fill a circular bottle or glass with constant amount of fluid. Plot the height of the fluid in the bottle at time $t$. Assume the radius of the bottle is $f(z)$ at height $z$. Can you find a formula for the height $g(t)$ of the water? This is not so easy. But we can find the rate of change $g'(t)$. Assume for example that $f$ is constant, then the rate of change is constant and the height of the water increases linearly like $g(t) = t$. If the bottle gets wider, then the height of the water increases slower. There is definitely a relation between the rate of change of $g$ and $f$. Before we look at this more closely, lets try to match the following cases of bottles with the graphs of the functions $g$ qualitatively.

**5** In each of the bottles, we call $g$ the height of the water level at time $t$, when filling the bottle with a constant stream of water. Can you match each bottle with the right height function?



a) b) d) c)



1) 2) 3) 4)

The key is to look at $g'(t)$, the rate of change of the height function. Because $[g(t+h)-g(t)]$ times the area $\pi f^2$ is a constant times the time difference $h = dt$, we have **calibration formula**

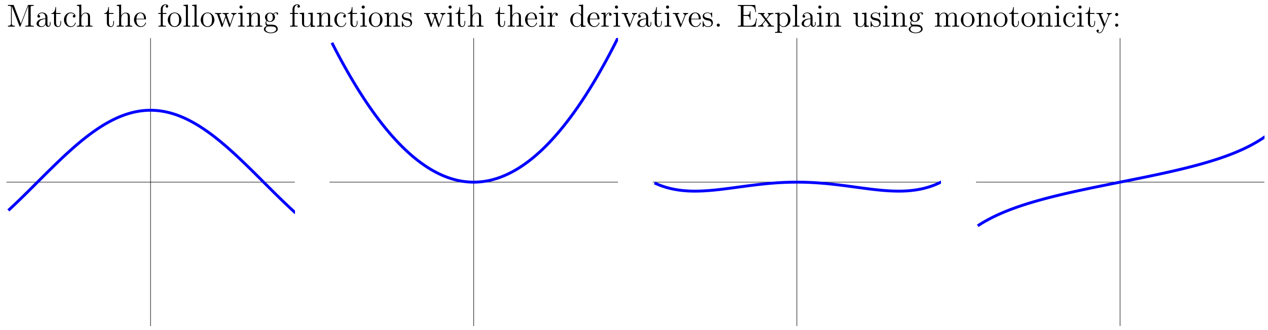
$$g' = \frac{1}{\pi f^2} \; .$$

It relates the derivative function of $g$ with the thickness $f(t)$ of the bottle at height $g$. No need to learn this. It just explains the story completely. It tells that that if the bottle radius $f$ is large, then the water level increase $g'$ is small and if the bottle radius $f$ is small, then the liquid level change $g'$ is large.

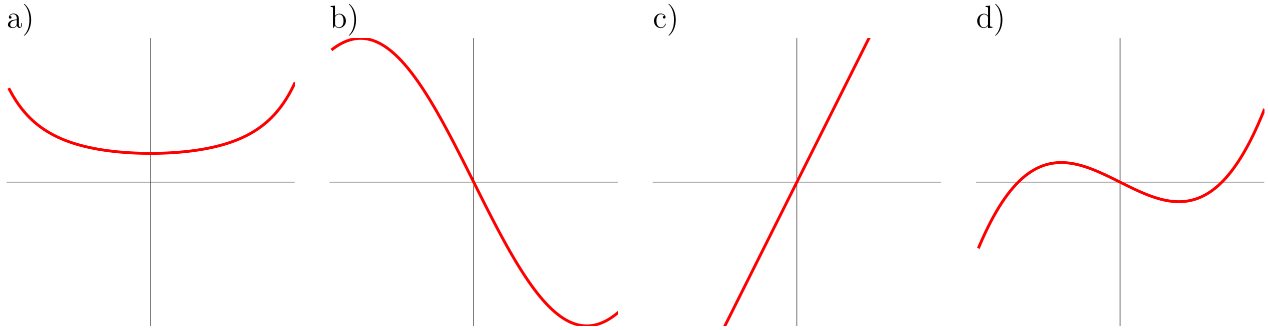
# Homework

1 For the following functions, determine on which intervals the function is monotonically increasing or monotonically decreasing.
a) $f(x) = -x^3 + x$ on $[-2, 2]$
b) $f(x) = \sin(x)$ on $[-2\pi, 2\pi]$
c) $f(x) = -x^4 + 8x^2$ on $[-4, 4]$.

2 Match the following functions with their derivatives. Explain using monotonicity:

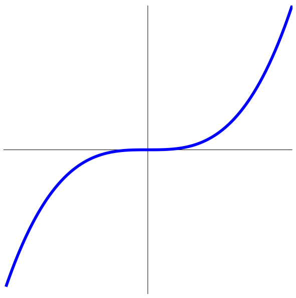


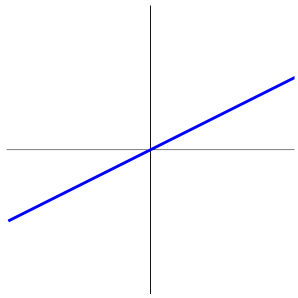
a) b) c) d)

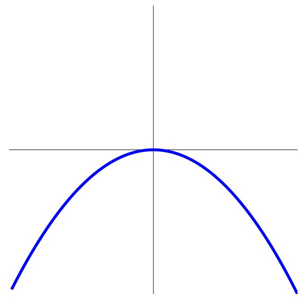


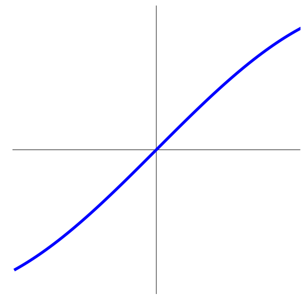
1) 2) 3) 4)

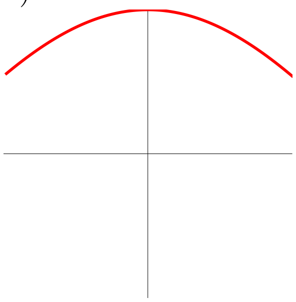
3 Match also the following functions with their derivatives. Give short explanations documenting your reasoning in each case.

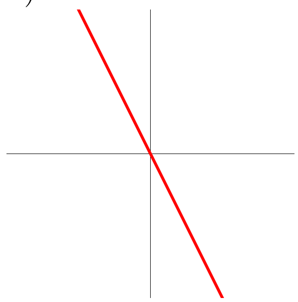

a)

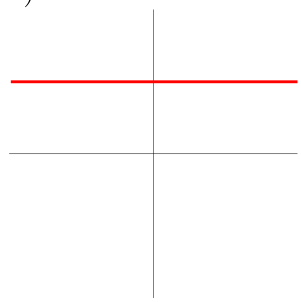

b)

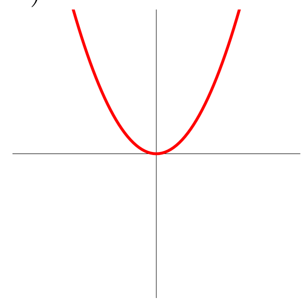

c)


d)


1)


2)


3)


4)

**4** Draw for the following functions the graph of the function $f(x)$ as well as the graph of its derivative $f'(x)$. You do not have to compute the derivative analytically as a formula here since we do not have all tools yet to compute the derivatives. The derivative function you draw needs to have the right qualitative shape however.

a) The **"To whom the bell tolls"** function

$$f(x) = e^{-x^2}$$

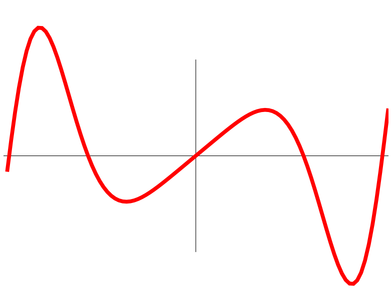
b) The **"witch of Maria Agnesi"** function:

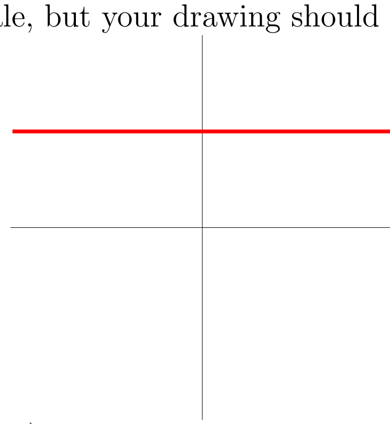
$$f(x) = \frac{1}{1+x^2}$$

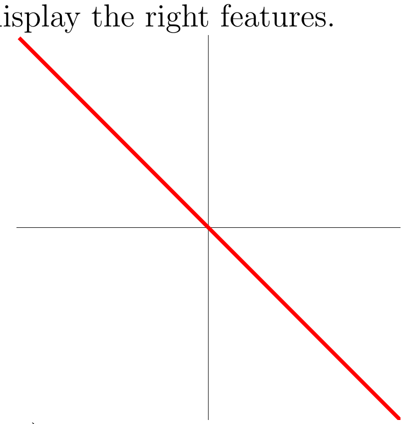
c) The **three gorges function**

$$f(x) = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} .$$

**5** Below you the graphs of three derivative functions $f'(x)$. In each case you are told that $f(0) = 1$. Your task is to draw the function $f(x)$ in each of the cases a),b),c). Your picture does not have to be up to scale, but your drawing should display the right features.


a)


b)


c)