Lecture 7: Rate of change

Given a function f and $h > 0$, we can look at the new function

$$Df(x) = \frac{f(x+h) - f(x)}{h} .$$

It is the **rate of change** of the function with **step size** h. When changing x to $x+h$ and then $f(x)$ changes to $f(x+h)$. The quotient $Df(x)$ is "rise over run". In this lecture, we take the limit $h \to 0$ and derive the important formulas $\frac{d}{dx}x^n = nx^{n-1}$, $\frac{d}{dx}\exp(x) = \exp(x)$, $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$ which we have seen already before in a discrete setting.

1 You walk up a snow hill of height $f(x) = 30 - x^2$ meters. Assume you walk with a step size of $h = 0.5$ meters. You are at position $x = 3$. How much do you climb or descend when making an other step? We have $f(3) = 21$ and $f(3.5) = 17.75$. We have walked down 3.25 meters. How steep was the snow hill at this point? We have to divide the height difference by the walking distance: $-3.25/0.5 = -7.5$. This is the slope with that step size.

Today, we take the limit $h \to 0$:

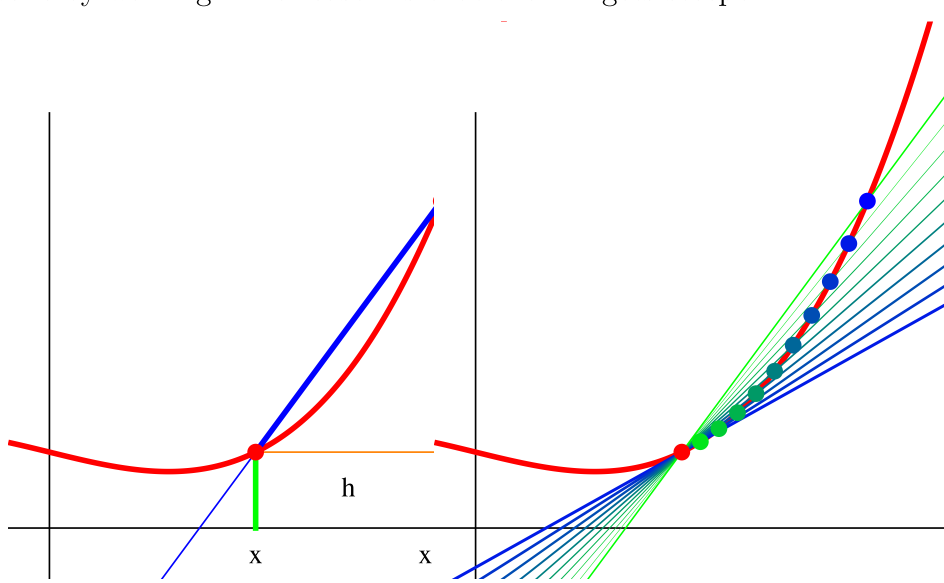
> If the limit $\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exist, we say f is **differentiable** at the point x. The value is called the **derivative** or **instantaneous rate of change** of the function f at x. We denote the limit also with $f'(x)$.

2 In the previous problem, $f(x) = 30 - x^2$ we have

$$f(x+h) - f(x) = [30 - (x+h)^2] - [30 - x^2] = -2xh - h^2$$

Dividing this by h gives $-2x - h$. The limit $h \to 0$ gives $-2x$. We have just seen that for $f(x) = x^2$, we get $f'(x) = -2x$. For $x = 3$, this is -6. The actual slope of the snow hill is a bit smaller than the estimate done by walking. The reason is that the hill gets steeper.

The derivative $f'(x)$ has a geometric meaning. It is the slope of the tangent at x. This is an important geometric interpretation. It is useful to think about x as "time" and the derivative as the rate of change of the quantity $f(x)$ in time.

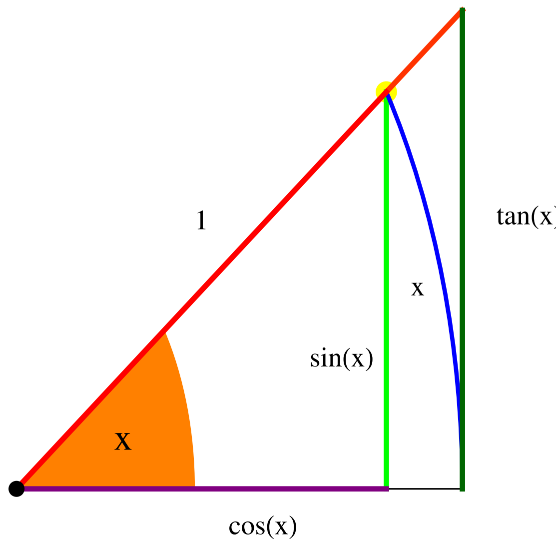


For $f(x) = x^n$, we have $f'(x) = nx^{n-1}$.

Proof: $f(x+h) - f(x) = (x+h)^n = (x^n + nx^{n-1}h + a_2h^2 + ... + h^n) - x^n = nx^{n-1}h + a_2h^2 + ... + h^n)$. If we divide by h, we get $nx^{n-1} + h(a_2 + \dots + h^{n-2})$ for which the limit $h \to 0$ exists: it is nx^{n-1}. This is an important result because most functions can be approximated very well with polynomials.

For $f(x) = \sin(x)$ we have $f'(0) = 1$ because the differential quotient is $[f(0+h) - f(0)]/h = \sin(h)/h = \mathrm{sinc}(h)$. We have already seen that the limit is 1 before. Lets look at it again geometrically. For all $0 < x < \pi/2$ we have

$$\sin(x) \leq x \leq \tan(x) \ .$$



3 [dividing by 2 squeezes the area of the sector by the area of triangles.] Because $\tan(x)/\sin(x) = 1/\cos(x) \to 1$ for $x \to 0$, the value of $\mathrm{sinc}(x) = \sin(x)/x$ must go to 1 as $x \to 0$. Renaming the variable x with the variable h, we see the **fundamental theorem of trigonometry**

$$\boxed{\lim_{h\to 0} \frac{\sin(h)}{h} = 1}$$

4 For $f(x) = \cos(x)$ we have $f'(x) = 0$. To see this, look at $f(0+h) - f(0) = \cos(h) - 1$. Geometrically, we can use Pythagoras $\sin^2(h) + (1-\cos(h))^2 \leq h^2$ to see that $2 - 2\cos(h) \leq h^2$ or $(1 - \cos(h) \leq h^2/2$ so that $(1 - \cos(h))/h \leq h/2$. This goes to 0 for $h \to 0$. We have just nailed down an other important identity

$$\boxed{\lim_{h\to 0} \frac{(1-\cos(h)}{h} = 0}.$$

The interpretation is that the tangent is **horizontal** for the cos function at $x = 0$. We will call this a critical point later on.

5 From the previous two examples, we get

$$\cos(x+h) - \cos(x) = \cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x) = \cos(x)(\cos(h) - 1) - \sin(x)\sin(h)$$

because $(\cos(h) - 1)/h \to 0$ and $\sin(h)/h \to 1$, we see that $[\cos(x+h) - \cos(x)]/h \to -\sin(x)$.

For $f(x) = \cos(ax)$ we have $f'(x) = -a\sin(ax)$.

6 Similarly,

$$\sin(x+h) - \sin(x) = \cos(x)\sin(h) + \sin(x)\cos(h) - \sin(x) = \sin(x)(\cos(h) - 1) + \cos(x)\sin(h)$$

because $(\cos(h) - 1)/h \to 0$ and $\sin(h)/h \to 1$, we see that $[\sin(x+h) - \sin(x)]/h \to \cos(x)$.

for $f(x) = \sin(ax)$, we have $f'(x) = a\cos(ax)$.

$$e = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

Like π, the Euler number e is irrational. Here are the first digits: 2.7182818284590452354. If you want to find an approximation, just pick a large n, like $n = 100$ and compute $(1+1/n)^n$. For $n = 100$ for example, we see $101^{100}/100^{100}$. We only need 101^{100} and then put a comma after the first digit to get an approximation. Interested why the limit exists: verify hat the fractions $A_n = (1+1/n)^n$ increase and $B_n = (1+1/n)^{(n+1)}$ decrease. Since $B_n/A_n = (1+1/n)$ which goes to 1 for $n \to \infty$, the limit exists. The same argument shows that $(1+1/n)^{xn} = \exp_{1/n}(x)$ increases and $\exp_{1/n}(x)(1+1/n)$ decreases. The limiting function $\exp(x) = e^x$ is called the **exponential function**. Remember that if we write $h = 1/n$, then $(1+1/n)^{nx} = \exp_h(x)$ considered earlier in the course. We can sandwich the exponential function between $\exp_h(x)$ and $(1+h)\exp_h(x)$:

$$\exp_h(x) \leq \exp(x) \leq \exp_h(x)(1+h), \quad x \geq 0 \; .$$

For $x < 0$, the inequalities are reversed.

7 Lets compute the derivative of $f(x) = e^x$ at $x = 0$. **Answer**. We have for $x \leq 1$

$$1 \leq (e^x - 1)/x \leq 1 + x \; .$$

Therefore $f'(0) = 1$. The exponential function has a graph which has slope 1 at $x = 0$.

8 Now, we can get the general case. It follows from $e^{x+h} - e^x = e^x(e^h - 1)$ that the derivative of $\exp(x)$ is $\exp(x)$.

For $f(x) = \exp(ax)$, we have $f'(x) = a\exp(ax)$.

It follows from the properties of taking limits that $(f(x) + g(x))' = f'(x) + g'(x)$. We also have $(af(x))' = af'(x)$. From this, we can now compute many derivatives

9 Find the slope of the tangent of $f(x) = \sin(3x) + 5\cos(10x) + e^{5x}$ at the point $x = 0$. **Solution**: $f'(x) = 3\cos(3x) - 50\sin(10x) + 5e^{5x}$. Now evaluate it at $x = 0$ which is $3 + 0 + 5 = 8$.

Finally, lets mention an example of a function which is not everywhere differentiable.

10 The function $f(x) = |x|$ has the properties that $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x =< 1$. The derivative does not exist at $x = 0$ evenso the function is continuous there. You see that the slope of the graph jumps discontinuously at the point $x = 0$.

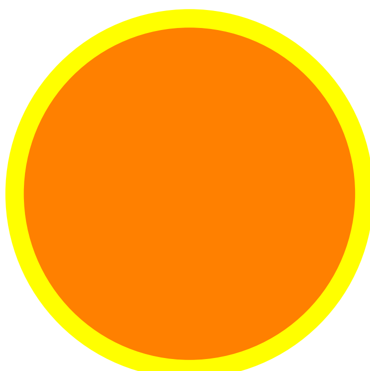
For a function which is discontinuous at some point, we don't even attempt to differentiate it there. For example, we would not even try to differentiate $\sin(4/x)$ at $x = 0$ nor $f(x) = 1/x^3$ at $x = 0$ nor $\sin(x)/|x|$ at $x = 0$. Remember these bad guys?

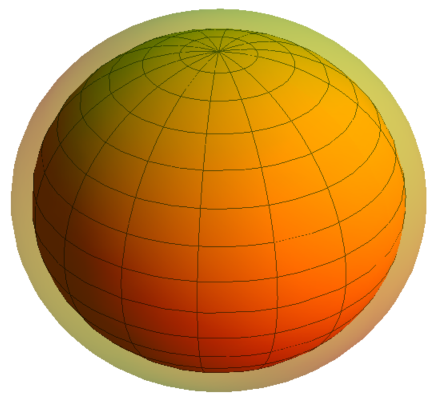
To the end, you might have noticed that in the boxes, more general results have appeared, where x is replaced by ax. We will look at this again but in general, the relation $f'(ax) = af(ax)$ holds ("if you drive twice as fast, you climb twice as fast").

Homework

1 For which of the following functions does the derivative $f'(x)$ exist for every x?
a) $|\sin(x)|$ b) $|\exp(x)|$ c) $x + \exp(7x) + 3\cos(45x)$
d) $\sin^4(x)$ e) $\sin(1/x)$ f) $\exp(-x) + |1 + \cos(15x)|$

2 a) A circle of radius x encloses a disc of area $f(r) = \pi r^2$. Find $\frac{d}{dr} f(r)$. Evaluate the rate of change at $r = 1/10$.
b) The ball of radius r has the volume $f(r) = 4\pi r^3/3$. Find $\frac{d}{dr} f(r))$ and compare it with the surface area of the sphere bounding the ball. Evaluate $\frac{d}{dr} f(r)$ at $r = 1/10$.
c) A **hypersphere** of radius r has the **hyper volume** $f(r) = \pi^2 r^4/2$. Find $\frac{d}{dr} f(r)$ and evaluate it at $r = 1/10$.





3 Find the derivatives of the following functions at the point $x = 0$.
a) $f(x) = 7\exp(4x) + \sin(11x) + x + x^4 + x^6 + x^{10}$.
b) $f(x) = (x^5 - 1)/(x - 1) + \cos(2x)$. First heal this function.
c) $f(x) = \frac{1+4x+6x^2+4x^3+x^4}{x^2+2x+1}$. Also here, first heal!

4 In this problem we compute the derivative of $\sqrt{x}$ for $x > 0$. To do so, we have to find the limit

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} .$$

Hint: multiply the top and the bottom with $(\sqrt{x+h} + \sqrt{x})$ and simplify.

5 On March 21, 2014, asteroid 2003 QQ47 will zoom by the earth about 20 million kilometers from earth. It is a rock of 1.24 km diameter which would make quite a firework if hitting the earth. By Pythagoras, its distance $y(t)$ from the earth is $\sqrt{t^2 + 20^2}$.
Let $f(t) = y(t)^2 = t^2 + 400$. What is the rate of change of $f(t)$ at $t = 0$? Explain. Does this mean the asteroid stops, when closest to the earth? See http://neo.jpl.nasa.gov/news/news138.html

Asteroid in 'near miss' with the Earth: Object capable of destroying London will pass within 17,000 miles from earth on Friday