Lecture 5: Intermediate Value Theorem

If $f(a) = 0$, then a is called a **root** of f. For $f(x) = \cos(2x)$ for example, there are roots of f at $x = \pi$ or $x = 3\pi$ or $x = -\pi$.

1 Find the roots of $f(x) = 4x + 6$. **Answer**: we set $f(x) = 0$ and solve for x. In this case $4x + 6 = 0$ and so $x = -3/2$.

2 Find the roots of $f(x) = x^2 + 2x + 1$. **Answer**: Because $f(x) = (x + 1)^2$ the function has a root at $x = -1$.

3 Find the roots of $f(x) = (x - 2)(x + 6)(x + 3)$. **Answer:** Since the polynomial is factored already, it is easy to see the roots $x = 2, x = -6, x = -3$.

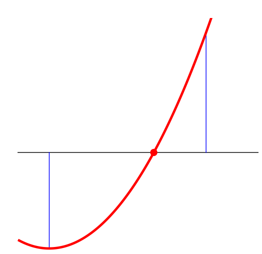
4 $f(x) = 12 + x - 13x^2 - x^3 + x^4$. Find the roots of f. While we do not have a formula for this, but we can try. Indeed, we see that $x = 1, x = -3, x = 4, x = -1$ are the roots.

5 The function $f(x) = \exp(x)$ does not have any root.

6 The function $f(x) = \log|x| = \ln|x|$ has roots $x = 1$ and $x = -1$.

7 $f(x) = 2^x - 16$ has the root $x = 2$.

Intermediate value theorem of Bolzano. If f is continuous on the interval $[a, b]$ and $f(a), f(b)$ have different signs, then there is a root of f in (a, b).

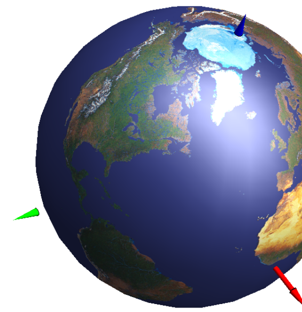


The proof is constructive: we can assume $f(a) < 0$ and $f(b) > 0$. The other case is similar. Look at the point $c = (a + b)/2$. If $f(c) < 0$, then look take $[c, b]$ as your new interval, otherwise, take $[a, c]$. We get a new root problem on a smaller interval. Repeat the procedure. After n steps, the search is narrowed to an interval $[u_n, v_n]$ of length $2^{-n}(b - a)$. Continuity assures that $f(u_n) - f(v_n) \to 0$ and $f(u_n), f(v_n)$ have different signs. Both u_n, v_n converge to a root of f.

8 Verify that the function $f(x) = x^{17} - x^3 + x^5 + 5x^7 + \sin(x)$ has a root. **Solution.** The function goes to $+\infty$ for $x \to \infty$ and to $-\infty$ for $x \to -\infty$. We have for example $f(10000) > 0$ and $f(-1000000) < 0$. The intermediate value theorem assures there is a point where $f(x) = 0$.

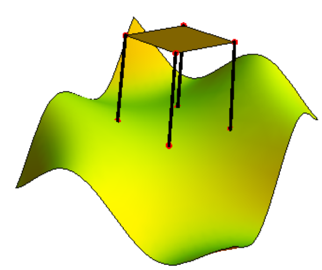
9 There is a solution to the equation $x^x = 10$. Solution: for $x = 1$ we have $x^x = 1$ for $x = 10$ we have $x^x = 10^{10} > 10$. Apply the intermediate value theorem.

10 **Earth Theorem.** There is a point on the earth, where temperature and pressure agrees with the temperature and pressure on the antipode.



Proof. Lets draw a meridian through the north and south pole and let $f(x)$ be the temperature on that circle. Define $g(x) = f(x) - f(x + \pi)$. If this function is zero on the north pole, we have found our point. If not, $g(x)$ has different signs on the north and south pole. There exists therefore an x, here the temperature is the same. Now, for every meridian, we have a latitude value $l(x)$ for which the temperature works. Now define $h(x) = l(x) - l(x + pi)$. This function is continuous. Start with meridian 0. If $h(0) = 0$ we have found our point. If not, then $h(0)$ and $h(\pi)$ take different signs. By the intermediate value theorem again, we have a root of h. At this point both temperature and pressure are the same than on the antipode. Remark: this argument in the second part is not yet complete. Do you see where the problem is?

11 **Wobbly Table Theorem.** On an arbitrary floor, a square table can be turned so that it does not wobble any more.



Proof. The 4 legs ABCD are on a square. Let x be the angle of the line AC with with some coordinate axes if we look from above. Given x, we can position the table **uniquely** as follows: the center of ABCD is on the z-axes, the legs ABC are on the floor and AC points in the direction x. Let $f(x)$ denote the height of the fourth leg D from the ground. If we find an angle x such that $f(x) = 0$, we have a position where all four legs are on the ground. Assume $f(0)$ is positive. (If it is negative, the argument is similar.) Tilt the table around the line AC so that the two legs B,D have the same vertical distance h from the ground. Now translate the table down by h. This does not change the angle x nor the center of the table. The two previously hovering legs BD now touch the ground and the two others AC are below. Now rotate around BD so that the third leg C is on the ground. The rotations and lowering procedures have not changed the location of the center of the table nor the direction. This position is the same as if we had turned the table by $\pi/2$. Therefore $f(\pi/2) < 0$. The intermediate value theorem assures that f has a root between 0 and $\pi/2$.

Lets call $Df(x) = (f(x+h) - f(x))/h$ the **discrete derivative** of f for the constant h. We will study it more in the next lecture. You have in a homework already verified that $D\exp_h(x) = \exp_h(x)$.

Lets call a point p, where $Df(x) = 0$ a **discrete critical point** for h. Lets call a point a a **local maximum** if $f(a) \geq f(x)$ in an open interval containing a. Define similarly a **local minimum** as a point where $f(a) \leq f(x)$.

12 The function $f(x) = x(x-h)(x-2h)$ has the derivative $Df(x) = 3x(x-h)$ as you have verified in the case $h = 1$ in the first lecture of this course in a worksheet. We will write $[x]^3 = x(x-h)(x-2h)$ and $[x]^2 = x(x-h)$. The computation just done tells that $D[x]^3 = 3[x]^2$. Since $[x]^2$ has exactly two roots $0, h$, the function $[x]^3$ has exactly 2 critical points.

13 More generally for $[x]^{n+1} = x(x-h)(x-2h)...(x-nh)$ we have $D[x]^{n+1} = (n+1)D[x]^n$. Because $[x]^n$ has exactly n roots, the function $[x]^{n+1}$ has exactly n critical points. Keep the formula

$$\boxed{D[x]^n = n[x]^{n-1}}$$

in mind!

14 The function $\exp_h(x) = (1+h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$. Because this function has no roots and the derivative is the function itself, the function has no critical points.

Critical points lead to extrema as we will see later in the course. In our discrete setting we can say:

Fermat's maximum theorem If f is continuous and has a critical point a for h, then f has either a local maximum or local minimum inside the open interval $(a, a+h)$.

Look at the range of the function f restricted to $[a, a+h]$. It is a bounded interval $[c, d]$ by the intermediate value theorem. There exists especially a point u for which $f(u) = c$ and a point v for which $f(v) = d$. These points are different if f is not constant on $[a, a+h]$. There is therefore one point, where the value is different than $f(a)$. If it is larger, we have a local maximum. If it is smaller we have a local minimum.

15 **Problem.** Verify that a cubic polynomial has maximally 2 critical points. **Solution** $f(x) = ax^3 + bx^2 + cx + d$. Because the x^3 terms cancel in $f(x+h) - f(x)$, this is a quadratic polynomial. It has maximally 2 roots.

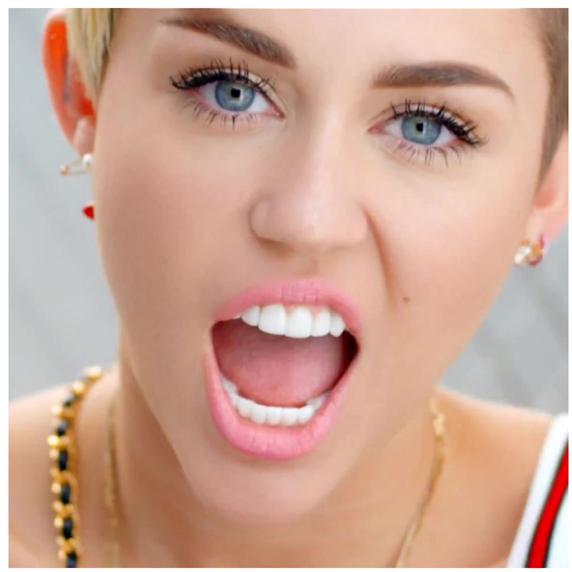
What we have called "critical point" here will in the limit $h \to 0$ be called "critical point" later in this course. While the h-critical point notion makes sense for any continuous function, we will need more regularity to take the limit $h \to 0$. This limit $h \to 0$ will be one of the major features.

Homework

1 Find the roots for $x^4 - 4x^3 - 7x^2 + 22x + 24$. You are told that all roots are integers.

2 Use the intermediate value theorem to verify that $f(x) = x^5 - 6x^4 + 8$ has at least two roots on $[-2, 2]$.

3 Miley' height is 165 cm. Gaga's height is 155 cm. Gaga was born March 28, 1986, Miley was born November 23, 1992. Assume Gaga owns now 500 millions and Miley owns 150 millions.
a) Can you argue that there was a moment when Miley's height was exactly half of Gaga's height?
b) Can you argue that there was a moment when Miley's fortune was exactly a tenth of Gagas fortune?
c) Many of you live in New York. Show that if you drive the 190 miles from here to New York in 4 hours then there are at least two moments of time when you drive with exactly 40 miles per hours. The trip is not part of a larger trip. Your start is in Boston and your Destination is New York.



4 Argue why there is a solution to
a) $5 + \sin(x) = x$.
b) $\exp(3x) = x$.
c) $\text{sinc}(x) = x^4$.
d) Why does the following argument not work:
The function $f(x) = 1/\cos(x)$ satisfies $f(0) = 1$ and $f(\pi) = -1$. There exists therefore a point x where $f(x) = 0$.
e) Does the function $f(x) = x + \log|\log|x||$ have a root somewhere?

5 a) Find a concrete function which has three local maxima and two local minima.
b) Let $h = 1$. Find a critical point for the function $f(x) = |x|$.
c) Verify that for any $h > 0$, the function $f(x) = x^5$ has no critical point in the sense given in the text: $[f(x + h) - f(x)]/h = 0$ is not possible.