Lecture 4: Continuity

A **function** f is called **continuous** at a point x_0 if a value $f(x_0)$ can be found such that $f(x) \to f(x_0)$ for $x \to x_0$. A function f is called **continuous on** $[a, b]$ if it is continuous for every point x in the interval $[a, b]$.

In the interior (a, b), the limit needs to exist both from the right and from the left. At the boundary a, only the right limit needs to exist and at b, only the left limit. Intuitively, a function is continuous if you can **draw the graph of the function without lifting the pencil**. Continuity means that small changes in x results in small changes of $f(x)$.

1 Any polynomial as well as $\cos(x), \sin(x), \exp(x)$ are continuous everywhere. Also the sum and product of such functions is continuous. For example

$$\sin(x^3 + x) - \cos(x^5 + x^3)$$

is continuous everywhere. We can also compose functions like $\exp(\sin(x))$ and still have a continuous function.

2 The function $f(x) = 1/x$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous. This discontinuity is called a **pole**. The **division by zero** kills continuity. Remember however that this can be salvaged in some cases like $f(x) = \sin(x)/x$ which is continuous everywhere. We consider the function healed at 0 even so it was at first not defined at $x = 0$.

3 The function $f(x) = \log|x|$ is continuous for x different from 0. It is not continuous at 0 because $f(x) \to -\infty$ for $|x| \to 0$. Keep the two examples, $1/x$ and $\log|x|$ in mind.

4 The function $\csc(x) = 1/\sin(x)$ is not continuous at $x = 0, x = \pi, x = 2\pi$ and any multiple of π. It has poles there because $\sin(x)$ is zero there and because we would divide by zero at such points. The function $\cot(x) = \cos(x)/\sin(x)$ shares the same discontinuity points as $\csc(x)$.

5 The function $f(x) = \sin(\pi/x)$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous due to **oscillation**. We can approach $x = 0$ in ways that $f(x_n) = 1$ and such that $f(z_n) = -1$. Just chose $x_n = 2/(4k + 1)$ or $z_n = 2/(4k - 1)$.

6 The **signum function** $f(x) = \text{sign}(x) = \left[\begin{array}{ll} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{array}\right.$ is not continuous at 0. It is a prototype of a function with a **jump** discontinuity at 0. It is impossible to heal the gap.

We can refine the notion of continuity and say that a function is **continuous from the right**, if there exists a limit from the right $\lim_{x \downarrow a} f(x) = b$. Similarly a function f can be continuous from the left only. Most of the time we mean with "continuous"= "continuous everywhere on the real line".

Rules:
a) If f and g are continuous, then $f + g$ is continuous.
b) If f and g are continuous, then $f * g$ is continuous.
c) If f and g are continuous and if $g > 0$ then f/g is continuous.
d) If f and g are continuous, then $f \circ g$ is continuous.

7 $\sqrt{x^2+1}$ is continuous everywhere on the real line.

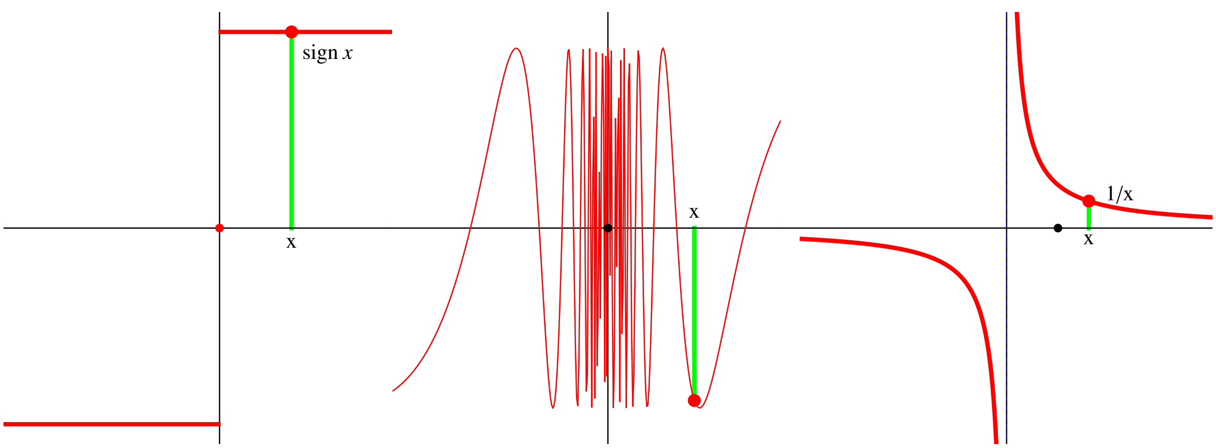
8 $\cos(x) + \sin(x)$ is continuous everywhere.

9 The function $f(x) = \log(|x|)$ (we write $\log = \ln$ for the natural log) is continuous everywhere except at 0. Indeed since for every integer n, we have $f(e^{-n}) = -n$, this can become arbitrarily large for $n \to \infty$ even so e^{-n} converges to 0 for n running to infinity.

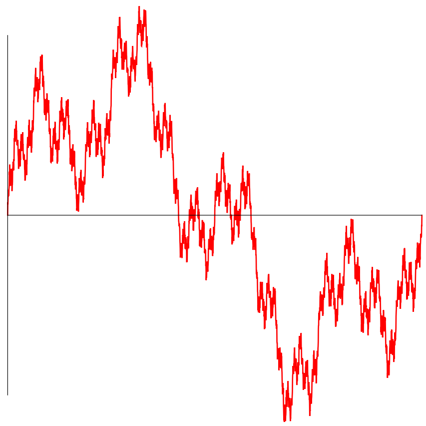
10 While $\log(|x|)$ is not continuous at $x = 0$, the function $1/\log|x|$ is continuous at $x = 0$. Is it continuous everywhere?

11 The function $f(x) = [\sin(x+h) - \sin(x)]/h$ is continuous for every $h > 0$. We will see next week that nothing bad happens when h becomes smaller and smaller and that the continuity will not deteriorate. Indeed, we will see that we get closer and closer to the cos function.

There are three major reasons, why a function is not continuous at a point: it can **jump**, **oscillate** or **escape** to infinity. Here are the prototype examples. We will look at more during the lecture.

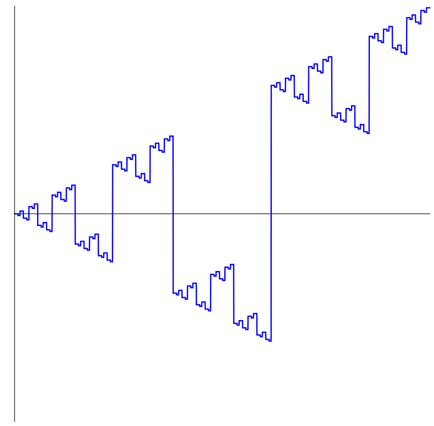


Why do we like continuity? We will see many reasons during this course but for now lets just say that:



A wild continuous function. This Weierstrass function is believed to be a fractal.

"Continuous functions can be pretty wild, but not too crazy."



A crazy discontinuous function. It is discontinuous at every point and known to be a fractal.

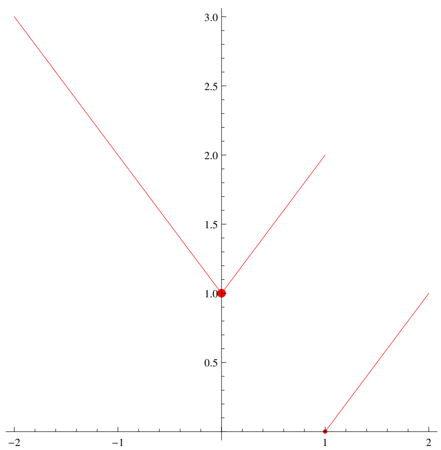
Continuity will be useful when finding maxima and minima. A continuous function on an interval $[a, b]$ has a maximum and minimum. We will see in the next hour that if a continuous function is negative at some place and positive at an other, there is a point between, where it is zero. Being able to find solutions to equations $f(x) = 0$ is important and much more difficult, if f not continuous.

12 **Problem** Determine for each of the following functions, where discontinuities appear:
a) $f(x) = \log(|x^2 - 1|)$
b) $f(x) = \sin(\cos(\pi/x))$
c) $f(x) = \cot(x) + \tan(x) + x^4$
d) $f(x) = x^4 + 5x^2 - 3x + 4$
e) $f(x) = \frac{x^2-4x}{x}$
Solution.
a) $\log(|x|)$ is continuous everywhere except at $x = 0$. Since $x^2 - 1 = 0$ for $x = 1$ or $x = -1$, the function $f(x)$ is continuous everywhere except at $x = 1$ and $x = -1$.
b) The function π/x is continuous everywhere except at $x = 0$. Therefore $\cos(\cos(\pi/x)$ is continuous everywhere except possibly at $x = 0$. We have still to investigate the point $x = 0$ but there, the function $\cos(\pi/x)$ takes values between -1 and 1 for points arbitrarily close to $x = 0$. The function $f(x)$ takes values between $\sin(-1)$ and $\sin(1)$ arbitrarily close to $x = 0$. It is not continuous there.
c) The function x^4 is continuous everywhere. The function $\tan(x)$ is continuous everywhere except at the points $k\pi$. The function $\cot(x)$ is continuous everywhere except at points $\pi/2 + k\pi$. The function f is therefore continuous everywhere except at the point $x = k\pi/2$, multiples of $\pi/2$.
d) The function is a polynomial. We know that polynomials are continuous everywhere.
e) The function is continuous everywhere except at $x = 0$, where we have to look at the function more closely. But we can heal the function by dividing nominator and denominator by x which is possible for x different from 0. The healed function is $f(x) = x - 4$.

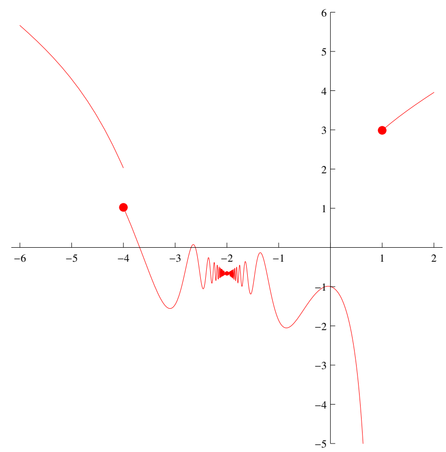
Homework

1 For the following functions, determine the points, where f is not continuous.
a) $\text{sinc}(x) + 1/\cos(x)$
b) $\sin(\tan(x))$
c) $f(x) = \cot(2 - x)$
d) $\text{sign}(x)/x$
e) $\frac{x^2+5x+x^4}{x-3}$
State which kind of discontinuity appears.

2 On which intervals are the following functions continuous?



a)



b)

3 Either do the following three problems a),b),c):
a) Construct a function which has a jump discontinuity and an escape to infinity.
b) Find a function which has an oscillatory discontinuity and an escape to infinity.
c) Find a function which has a jump discontinuity as well as an oscillatory discontinuity.
or shoot down the problem with one strike:
Find a function which has a jump discontinuity, a pole and an oscillatory discontinuity all at the same time.

4 Heal the following functions to make them continuous
a) $(x^3 - 8)/(x - 2)$
b) $(x^5 + x^3)/(x^2 + 3)$
c) $((\sin(x))^3 - \sin(x))/(\cos(x)\sin(x))$.
d) $(x^4 + 4x^3 + 6x^2 + 4x + 1)/(x^3 + 3x^2 + 3x + 1)$
e) $(x^{70} - 1)/(x^{10} - 1)$

5 Are the following function continuous? Break the functions up into simpler functions and analyze each. If you are not sure, experiment by plotting the functions.
a) $\sin(\frac{1}{3+\sin(x)\cos(x)}) + |\cos(x)| + \frac{\sin(x)}{x} + x^5 + x^3 + 1 + \frac{7}{\exp(x)}$.
b) $\frac{2}{\log|x|} + x^7 - \cos(\sin(\cos(x))) - \exp(\log(\exp(x)))$