Lecture 1: What is Calculus?

Calculus generalizes the processes of **taking differences** and **performing summation**. Differences measure **change**, sums quantify how things **accumulate**. The process of taking differences has a limit called **derivative**. The process of taking sums has a limit called **integral**. These two operations are related in an intimate way. In this first lecture, we look at calculus in the simplest possible setup. Functions are evaluated on integers and no limits are taken. About 20'000 years ago, numbers were represented by units like

$$1, 1, 1, 1, 1, 1, \dots .$$

Numbers were carved into the fibula bone of a baboon like **Ishango bone**. [1]. It took thousands of years until humans figured out to work with numbers using symbols like

$$0, 1, 2, 3, 4, \dots .$$

The concept of function came much, much later and allows to write the counting process as $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 3$. We mean with a **function** f a rule that assigns to an input like 1001 an output like $f(1001) = 1001$. Lets look at $Df(n) = f(n+1) - f(n)$, the **difference** between two function values. It is a function again. The counting function f we have just considered satisfies $Df(n) = 1$ for all n. We can also formalize the summation process. If $g(n) = 1$ is the function which assigns the constant value 1 for all n, then $Sg(n) = g(0) + g(1) + \dots + g(n-1) = 1 + 1 + \dots + 1 = n$. We see that $Df = g$ and $Sg = f$. Lets start with $f(n) = n$ and apply the **summation process** on that function:

$$Sf(n) = f(0) + f(1) + f(2) + \dots + f(n-1) .$$

We get the following values:

$$0, 1, 3, 6, 10, 15, 21, \dots .$$

The new function g satisfies $g(1) = 1, g(2) = 3, g(3) = 6$, etc. These numbers are called **triangular numbers**. From the function g we can get f back by taking differences:

$$Dg(n) = g(n+1) - g(n) = f(n) .$$

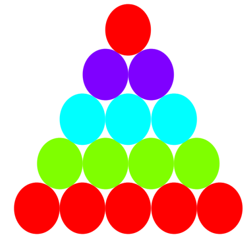
For example $Dg(5) = g(6) - g(5) = 15 - 10 = 5$. And indeed this is $f(5)$. Finding a formula for the sum $Sf(n)$ is not so easy. Can you do it?

Legend tells that when **Karl-Friedrich Gauss** was a 9 year old school kid, his teacher, Mr. Büttner gave him the task to sum up the first 100 numbers $1 + 2 + \dots + 100$. Gauss discovered that pairing the numbers up would simplify the summation He would write the sum as $(1+100)+(2+99)+\dots+(50+51)$ so that the answer is $g(n) = n(n-1)/2 = 5050$. He got so an explicit expression for the sum function without actually doing the sum. Lets apply the difference function again: $Dg(n) = n(n+1)/2 - n(n-1)/2 = n = f(n)$.

[1] The lengths of the marks might have had astronomical significance but this is not relevant here

Since this went so well, lets add up the new sequence again and compute $h = Sg$. We get the sequence

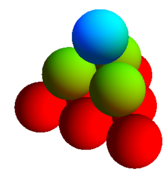
$$0, 1, 4, 10, 20, 35, \ldots .$$



These numbers are called the **tetrahedral numbers**. The reason is that one can use $h(n)$ balls to build a tetrahedron of side length n. For example, we need $h(4) = 20$ golf balls to build a tetrahedron of side length 4. The formula which holds for h is $\boxed{h(n) = n(n-1)(n-2)/6}$. In the worksheet we will check that summing the differences gives the function back. What we have seen in special cases is a general theorem:

$$\boxed{SDf(n) = f(n) - f(0), \qquad DSf(n) = f(n)}$$

It is an arithmetic version of the **fundamental theorem of calculus**. Our goal is to see how this works when we take limits. We will see the **integral** $\boxed{\int_0^x f(x)\, dx}$ as well as the **derivative** $\boxed{\frac{d}{dx} f(x)}$ and verify the **fundamental theorem of calculus**:

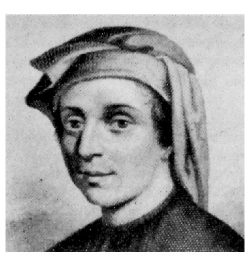


$$\boxed{\int_0^x \frac{d}{dt} f(t)\, dt = f(x) - f(0), \qquad \frac{d}{dx} \int_0^x f(t)\, dt = f(x)}$$

It is a fantastic result. A major goal of this course will be to understand this result and see how to apply it. We will revisit the theorem at least twice again. [2] The above version will lead us. Note that if we define $[n]^0 = 1, [n]^1 = n, [n]^2 = n(n-1)/2, [n]^3 = n(n-1)(n-2)/6$ then $D[n] = [1], D[n]^2 = 2[n], D[n]^3 = 3[n]^2$ and in general

$$\boxed{\frac{d}{dx}[x]^n = n[x]^{n-1}}$$

We will generalize this from $h = 1$ to general $h > 0$ and then see that it also holds in the limit $h \to 0$, where it becomes the familiar formula $(d/dx)x^n = nx^{n-1}$ you might know already. The calculus you have just seen, contains the essence of single variable calculus. This core idea will become more powerful and natural if we use it together with the concept of limit.

1 

Problem: The sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ satisfies the rule $f(x) = f(x-1) + f(x-2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$, if we assume $f(0) = 0$? **Solution:** We take the difference between successive numbers and get the sequence of numbers

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

which is the same sequence again. We can deduce from this recursion that f has the property that $\boxed{Df(x) = f(x-1)}$. It is called the **Fibonnacci sequence**, a sequence of great fame.

2 **Problem:** Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first n numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, \ldots$. What sequence is that?
Solution: Because $Df(x) = f(x-1)$ we have $f(x) - f(0) = SDf(x) = Sf(x-1)$ so that $Sf(x) = f(x+1) - f(1)$. Summing the Fibonnacci sequence produces the Fibonnacci sequence shifted to the left with $f(2) = 1$ is subtracted. It has been relatively easy to find the sum, because we knew what the difference operation did. This example shows:

[2] Many textbooks need hundreds of pages until the fundamental theorem is reached.

We can study differences to understand sums.

The next problem illustrates this too:

3 **Problem:** Find the next term in the sequence
2 6 12 20 30 42 56 72 90 110 132 . **Solution:** Take differences

2	6	12	20	30	42	56	72	90	110	132
2	4	6	8	10	12	14	16	18	20	22
2	2	2	2	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	0

Now we can add an additional number, starting from the bottom and working us up.

2	6	12	20	30	42	56	72	90	110	132	156
2	4	6	8	10	12	14	16	18	20	22	24
2	2	2	2	2	2	2	2	2	2	2	2
0	0	0	0	0	0	0	0	0	0	0	0

4 **Problem:** The function $f(n) = 2^n$ is called the **exponential function**. We have for example $f(0) = 1, f(1) = 2, f(2) = 4, \dots$. It leads to the sequence of numbers

n=	0	1	2	3	4	5	6	7	8	...
f(n)=	1	2	4	8	16	32	64	128	256	...

We can verify that f satisfies the equation $\boxed{Df(x) = f(x)}$. because $Df(x) = 2^{x+1} - 2^x = (2-1)2^x = 2^x$.
This is an important special case of the fact that

The derivative of the exponential function is the exponential function itself.

The function 2^x is a special case of the exponential function when the Planck constant is equal to 1. We will see that the relation will hold for any $h > 0$ and also in the limit $h \to 0$, where it becomes the classical exponential function e^x which plays an important role in science.

5 **Problem:** Look at the function $f(n)$ which gives the n'th prime number. Lets look at the derivatives $D^k f$ but take the absolute value $|D^k(f)|$. In other words, we study $T(f)(n) = |f(n+1) - f(n)|$. We know for example that $f(n) = 2^n$ satisfies $Tf = f$. Lets look at the prime function and apply this differences:

n=	1	2	3	4	5	6	7	8	9	...
$f(n) =$	2	3	5	7	11	13	17	23	29	...
$Tf(n) =$	1	2	2	4	2	4	2	4	6	...
$T^2 f(n) =$	1	0	2	2	2	2	2	2	4	...
$T^3 f(n) =$	1	2	0	0	0	0	0	2	0	...

The **Gilbreath conjecture** of 1959 claims that the first entry remains 1 for ever when applying this absolute differentiation process. The problem is still open.

Homework

1 Predict the future and find the next term in the sequence

$$2, 10, 30, 68, 130, 222, 350, 520, 738, 1010, 1342, \ldots$$

by taking "derivatives" and then by "integrating".

2 Look at the odd numbers $f(n) = 2n + 1$. The sequence starts with

$$1, 3, 5, 7, 9, 11, 13, \ldots .$$

Play around and compute $Sf(1) = 1, Sf(2) = 1 + 3, Sf(3) = 1 + 3 + 5$. Guess a formula for $Sf(n) = f(0) + f(1) + f(2) + \ldots + f(n-1)$. Verify that $g(n+1) - g(n) = f(n)$ by plugging in your formula for g and simplifying.

3 We have defined $Sf(n) = f(0) + f(1) + f(2) + ... + f(n-1)$ and $Df(n) = f(n+1) - f(n)$ and seen the first two lines of the following table:
(The notation $\boxed{[n]^k}$ is defined on the left of the following table. It is notation so that $D[n]^k = k[n]^{k-1}$ holds.)

$f(n) = [n]^0 = 1$	we have	$g(n) = Sf(n) = n$.
$f(n) = [n]^1 = n$	we have	$g(n) = Sf(n) = n(n-1)/2$.
$f(n) = [n]^2/2 = n(n-1)/2$	we have	$g(n) = Sf(n) = n(n-1)(n-2)/6$.

Verify the third line. That is, show $g(n) = [n]^3/6 = n(n-1)(n-2)/6$ satisfies $Dg(n) = f(n) = [n]^2/2 = n(n-1)/2$.

4 Find $F(n) = Sf(n)$ for the function $f(n) = n^2$. This means we want to find a formula such that $F(1) = 1, F(2) = 5, F(3) = 14$ leading to the sequence of numbers

$$0, 1, 5, 14, 30, 55, 91, 140, 204, 285, \ldots .$$

We have already have computed Sg for $g(n) = n$ as well as Sh for $h(n) = n(n-1)/2$ in a previous problem. Try to write f as a combination of g and h and use the **addition rule** $D(g + h) = Dg + Dh$.

5 Find a formula for $g(n) = Df(n)$ for the function $f(n) = 3^n$.
Extra credit (you do not have to turn this in). Can you use this to find a formula for $1 + 3 + 9 + 27 + ...$ Hint: use the fact that D and S are inverses of each other.

General remarks about homework

- Make sure to think about the problem yourself first before discussing it with others.
- The time you spend on homework is valuable. Especially the exploration time before you know how to solve it.
- If you do not know how to get started, don't hesitate to ask.