5/17/2014: Final Exam Practice E

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points) No justifications are needed.

1) T F $\frac{d}{dx} \log(\cos(x)) = -\tan(x)$.

Solution:
Differentiate using the chain rule. This integral appeared later in a main problem.

2) T F If $f'(x) > 0$ for all $x \geq 0$ and $f(0) = 1$ then $f(x) > 0$ for all $x \geq 0$.

Solution:
The function is monotonically increasing and will therefore even satisfy $f(x) \geq 1$ for all $x > 0$.

3) T F If a function $f(x)$ has a critical point 1 and $f''(1) > 0$ then 1 is a local maximum of f.

Solution:
It is a local minimum, not a local maximum.

4) T F The anti derivative of $\log(x)$ is $1/x$ for all $x > 0$.

Solution:
It is the derivative, not the anti derivative

5) T F The limit of $1/(\log|x|)$ for $x \to 0$ exists and is 0.

Solution:
As now covered several times, since $\log|x|$ goes to $-\infty$ for $x \to 0$, the limit exists.

6) T F Our musical 12 tone scale is based on the exponential function. The frequencies are $440 \cdot 2^{k/12}$, where k is an integer.

Solution:
Yes, we have seen this even more precisely as the midi function.

7) T F We have $\sin(9\pi/4) = -1/\sqrt{2}$.

Solution:
The sign is different. It is $1/\sqrt{2}$. The simplest way to see is to subtract 2π.

8) T F The function $t/(e^t - 1)$ has the limit 1 as t goes to zero.

Solution:
Use Hopital

9) T F If f and g are continuous functions on the real line, then both fg and f/g are continuous functions on the real line.

Solution:
g could be zero.

10) T F The point $x = 0$ is a critical point of $f(x) = x^2 - x$.

Solution:
Differentiate and see that $x = 2$ is the only one.

11) T F A function f for which $f'(x) < 0$ for all x is monotonically decreasing.

Solution:
Yes, that is it

12) T F The function $f(x) = \exp(x)$ has no inflection point.

Solution:
Its second derivative is everywhere positive.

13) T F The function $f(x) = (x^3 - 1)/(x - 1)$ has a limit for $x \to 1$.

Solution:
You can factor out $(x - 1)$ and get $x^2 + x^1$.

14) T F If the velocity at time t is $\sin(t)$ and $f(0) = 0$, then the position at time t is $f(t) = 1 - \cos(t)$.

15) T F If a function f is differentiable and $f(x) \to 0$ for $x \to \infty$, then $\int_1^\infty f(x)\, dx$ is bounded.

Solution:
The function $f(x) = \sin^2(x)/x$ is a counter example.

16) T F We have the differentiation rule $(fg)' = (f'g - fg')/g^2$ if g is not zero.

Solution:
The right hand side is the quotient rule.

17) T F Hopital's rule tells that if $f(x) \to 0$ for $x \to 0$ and $g(x) \to 0$ for $x \to 0$, we have $\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0} f'(x)g'(x)$.

Solution:
Hopital deals with the quotient and not with the product.

18) T F A Newton step for the function f is $T(x) = x - \frac{f(x)}{f'(x)}$.

Solution:
By definition

19) T F A catastrophe is a critical point of f.

Solution:
A catastrophe is a parameter value.

20) T F The fundamental theorem of calculus tells that $\int_0^x f'(t)\, dt = f(x) - f(0)$ and $\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$.

Solution:
Yes, this is the most important result in this course.

Problem 2) Matching problem (10 points) No justifications are needed.

a) (5 points) Name dropping: Match results with names

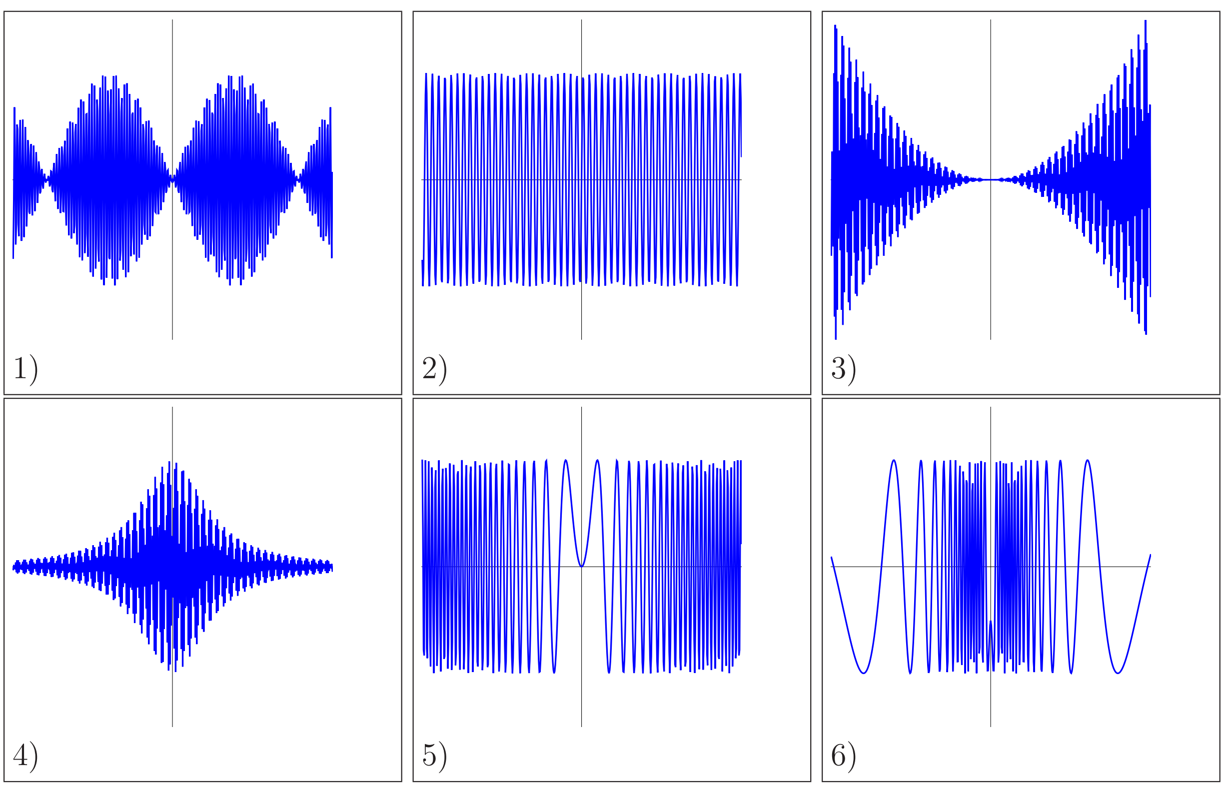
Result	Enter A-G
Fundamental theorem of trigonometry	
Newton steps	
Fundamental theorem of calculus	
Mean value theorem	
Rolle's theorem	
Intermediate value theorem	
Fermat theorem	

A)	$\int_0^1 f'(x)\,dx = f(1) - f(0)$
B)	$\lim_{x\to 0} \sin(x)/x = 1$
C)	$f(0) = -1, f(1) = 1$ implies $f(x) = 0$ for some $x \in (0,1)$.
D)	f is continuous on $[0,1]$ then f has a global max and min on $[0,1]$.
E)	$T(x) = x - f(x)/f'(x)$.
F)	If $f(0) = f(1) = 0$ then $f'(x) = 0$ for some $x \in (0,1)$.
G)	There exists x in $(0,1)$ such that $f'(x) = f(1) - f(0)$.

b) (5 points) Match the functions with their graphs. Mind the ceiling hulls!

Function	Enter 1-6
$\sin(1000x)$	
$\sin(1000x^2)$	
$x^2\sin(1000x)$	

Function	Enter 1-6
$\sin(1000x/(1+x^2))$	
$\sin(x)\sin(1000x)$	
$\sin(1000x)/(1+x^2)$	



Solution:
a) BEAGFCD
b) 2,5,3 and 6,1,4

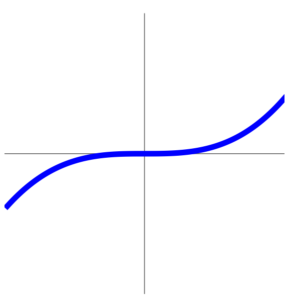
Problem 3) Matching problem (10 points) No justifications are needed.

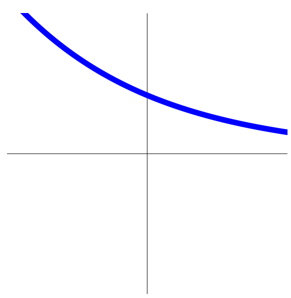
a) (5 points) Find the relation between the following functions:

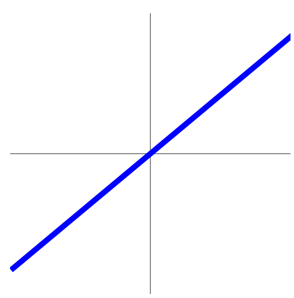
function f	function g	$f = g'$	$g = f'$	none
$\log(x)$	$1/x$			
$1/x$	$-1/x^2$			
$\tan(x)$	$1/(1+x^2)$			
$\cot(x)$	$-1/\sin^2(x)$			
$\arctan(x)$	$1/\cos^2(x)$			
$\text{arccot}(x)$	$-1/(1+x^2)$			

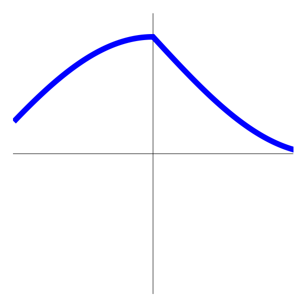
b) (5 points) Match the following functions (a-d) with their derivatives (1-4) and second derivatives (A-D).

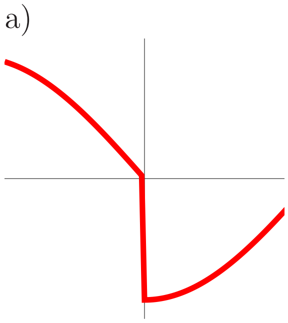
Function a)-d)	Fill in 1)-4)	Fill in A)-D)
graph a)		
graph b)		
graph c)		
graph d)		

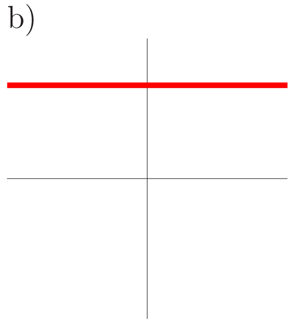

a)

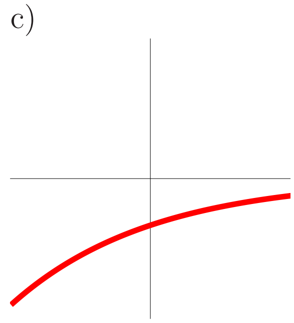

b)

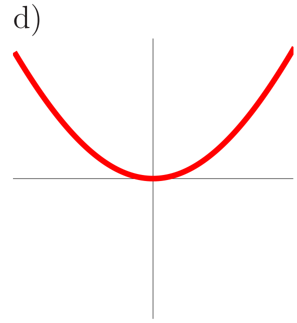

c)


d)

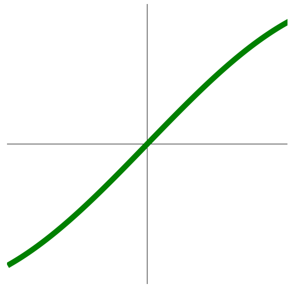

1)


2)

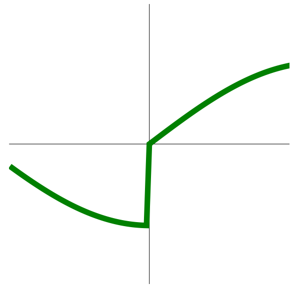

3)


4)

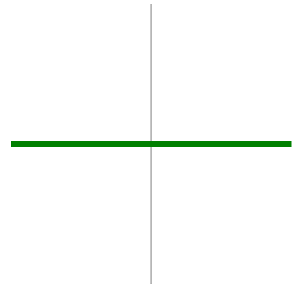
A)



B)



C)

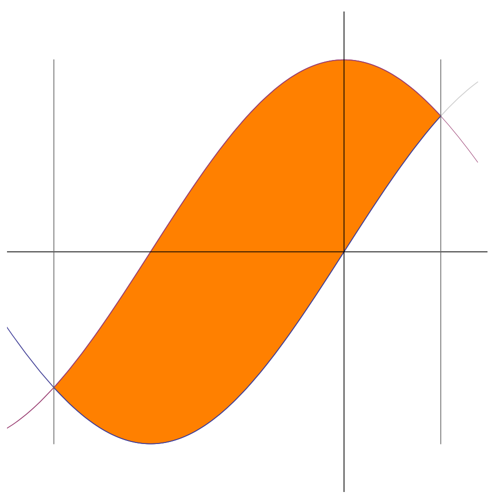


D)

Solution:
a) all are g=f' except 3nd and 5th, where nothing applies.
b)
4 B
3 A
2 D
1 C

Problem 4) Area computation (10 points)

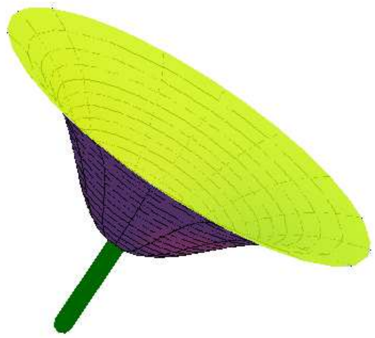
Find the area enclosed by the functions $f(x) = \sin(x)$ and $f(x) = \cos(x)$ and between the vertical lines $x = -3\pi/4$ and $x = \pi/4$.

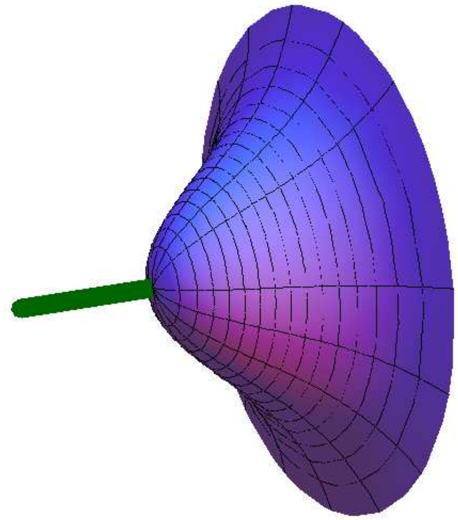


Solution:
$\int_{-3\pi/4}^{\pi/4} \cos(x) - \sin(x)\ dx = 4\sqrt{2}/2$ which is $\boxed{2\sqrt{2}}$.

Problem 5) Volume computation (10 points)

If we rotate the graph of the function $f(x) = \sqrt{\tan(x)}$ from $x = 0$ to $x = \pi/4$ we obtain a flower shaped solid. The two pictures below allow to admire it from two sides. Find its volume.





Solution:

$$\int_0^{\pi/4} \tan(x)\, dx = \int_0^{\pi/4} -\log(\cos(x))\, dx = \pi \log(\sqrt{2}/2) = \pi \log(2)/2 \; .$$

The integral by the has appeared in the first TF problem.

Problem 6) Improper integrals (10 points)

a) (5 points) Find the integral or state that it does not exist

$$\int_1^{\infty} \frac{1}{x^5}\, dx \; .$$

b) (5 points) Find the integral or state that it does not exist

$$\int_0^{\infty} \frac{1}{\cos^2(x)}\, dx \; .$$

Solution:

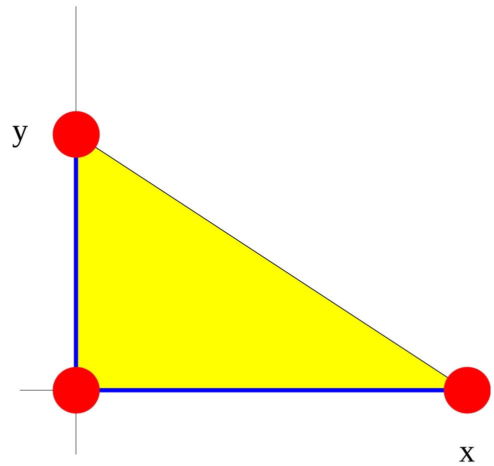
a) The antiderivative is $-(1/4)x^{-4}$. This is an improper integral but the limit exists: 1/4.

b) We have $\int_0^r 1/\cos^2(x)\; dx = \tan(x)_0^r$. This has no limit for $r \to \infty$ because tan is periodic.

Problem 7) Extrema (10 points)

a) (5 points) Find the local and global maxima of $f(x) = 2x^3 - 3x^2$ on the interval $[-1, 2]$. Use the second derivative test to check local extrema.

b) (5 points) Which rectangular triangle $(0,0), (x,0), (0,y)$ with $x + y = 2, x \geq 0, y \geq 0$ has maximal area $A = xy/2$?



Solution:
a) $f'(x) = 6x(x-1)$ has critical points at $0, 1$. From $f''(x) = 12x - 6$ we see that 0 is a local max and 1 is a local min. We also have to look at the boundary points $-1, 2$. Comparing the values of the function at $-1, 0, 1, 2$ shows that 2 is the global maximum and -1 is the global minimum.
b) Substitute y into the area formula to get a function $f(x) = x(2-x)/2$ of one variable. It has the derivative $1 - x$ which has a critical point at $x = 1$.

Problem 8) Integration by parts (10 points)

Find the antiderivative:

$$\int x^4 \sin(x-1) \, dx \, .$$

Solution:
Use the Tic-Tac-Toe integration method:

x^4	$\sin(x-1)$	
$4x^3$	$-\cos(x-1)$	$\oplus$
$12x^2$	$-\sin(x-1)$	$\ominus$
$24x$	$\cos(x-1)$	$\oplus$
24	$\sin(x-1)$	$\ominus$
0	$-\cos(x-1)$	$\oplus$

Collect things together $4x\,(x^2-6)\sin(x-1) - (x^4 - 12x^2 + 24)\cos(1-x)$.

Problem 9) Substitution (10 points)

a) (3 points) Solve the integral $\int \sin^4(x)\cos(x)\;dx$.

b) (4 points) Solve the integral $\int \sqrt{1+2x}\;dx$.

c) (3 points) Find the integral $\int \cos(x^2+1)x\;dx$.

Solution:
These are all standard substitution problems.
a) $\sin^5(x)/5 + C$.
b) $(1+2x)^{3/2}/3 + C$.
c) $\sin(x^2+1)/2 + C$

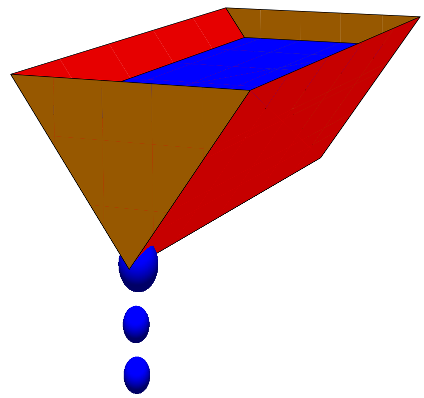
Problem 10) Partial fractions (10 points)

Find

$$\int_4^5 \frac{1}{(x-2)(x-3)}\;dx\;.$$

Solution:
$1/(x-3) - 1/(x-2) = 1/(x-3)(x-2)$. Now integrate: $\log|x-3| - \log|x-2||_4^5 = \log|4| - \log(3)$.

Problem 11) Related rates (10 points)

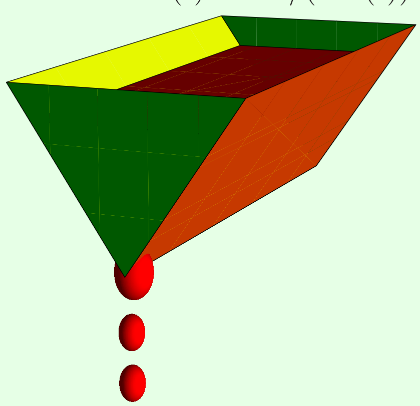


Water drips from warthog Tuk's bath tub, an uncomfortable container of length 10 for which the width is $2z$ at height z. The volume of the water filled up to height z is

$$V(z) = 10z^2 \; .$$

If the volume $V(z(t))$ decreases with constant rate $V' = -1$, how fast does the water level sink? Especially, what is $z'(t)$ at $t = 1$ if $V(1) = 10$.

Solution:
Tuk had first been destined been slaughtered so that this related rate problem would deal with blood dripping from the tub. But Tuk is such a nice warthog, that the farmer could not do it, so Tuk lived on happily ever after, soaking for the rest of his life every week in his triangular hot tub.
$-1 = V' = 20zz'$ shows that $z'(t) = -1/(20z(t))$. At $t = 1$ we have $z = 1$ and $-1/(20z(1))$ which is $-1/20$.



Problem 12) Various integration problems (10 points)

Find the anti-derivatives of the following functions:

a) (3 points) $f(x) = \log(x)/x$.

b) (4 points) $f(x) = \frac{1}{x^2-4}$.

c) (3 points) $f(x) = \frac{1}{x\log(x)}$.

Solution:
These problems were inspired partly by Pillow problems covered early in the course.
a) $\log(x)^2/2 + C$.
b) $[\log(x-2) - \log(2+x)]/4 + C$.
c) $\log(\log(x)) + C$.

Problem 13) Applications (10 points)

If $F(x) = \log(x)$ is the total cost and $f(x) = F'(x)$ is the marginal cost and $g(x) = \log(x)/x$ is the average cost:.

a) (4 points) Find the break-even point $f = g$.

b) (4 points) Where is the average cost maximal?

c) (2 points) Sweet surprise: why are the results in a) b) the same? The answer to c) is one word.

Solution:
a) $f = g$ for $1 = \log(x)$ meaning $x = e$.
b) $g' = (1 - \log(x))/x^2 = 0$ means $x = e$.
c) Strawberry!