5/17/2014: Final Exam

Your Name:

- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- All functions f if not specified otherwise can be assumed to be smooth so that arbitrary many derivatives can be taken.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- Except for multiple choice problems, give computations.
- No notes, books, calculators, computers, or other electronic aids are allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1) T F $\cos(17\pi/4) = \sqrt{2}/2$.

Solution:
Yes, it is $\cos(\pi/4)$.

2) T F The tangent function is monotonically increasing on the open interval $(-\pi/2, \pi/2)$.

Solution:
Indeed, its derivative is $1/\cos^2(x)$.

3) T F The arccot function is monotonically increasing from $\pi/4$ to $3\pi/4$.

Solution:
It is decreasing.

4) T F If f is a probability density function, then $\int_{-\infty}^{\infty} f(x)\, dx = 0$

Solution:
It is 1.

5) T F $\frac{d}{dx} e^{\log(x)} = 1$.

Solution:
First simplify.

6) T F If $f''(0) = -1$ then f has a local maximum at $x = 0$.

Solution:
We need the first derivative to be zero

7) T F The improper integral $\int_{-1}^{1} 1/|x|\, dx$ is finite.

Solution:
This is an improper integral which does not exist

8) T F The function $-\cos(x) - x$ has a root in the interval $(-100, 100)$.

Solution:
Use the intermediate value theorem.

9) T F If a function f has a local maximum in $(0, 1)$ then it also has a local minimum in $(0, 1)$.

Solution:
Take $f(x) = -x^2$. It does not have a local minimum.

10) T F The anti derivative of $1/(1 - x^2)$ is equal to $\arctan(x)$.

Solution:
It is an other sign.

11) T F The function $f(x) = (e^x - e^{2x})/(x - x^2)$ has the limit 1 as x goes to zero.

Solution:
Use Hopital's rule to see that it is the same than the limit $(e^x - 2e^{2x})/(1 - 2x)$ for for $x \to 0$ but this is -1.

12) T F If you listen to the sound $e^{-x}\sin(10000x)$, then it gets louder and louder as time goes on.

Solution:
The amplitude decays like e^{-x}.

13) T F The function $f(x) = e^{x^2}$ has a local minimum at $x = 0$

Solution:
The function is positive near 0 but equal to zero at 0.

14) T **F** The function $f(x) = (x^{55} - 1)/(x - 1)$ has the limit 1 for $x \to 1$.

Solution:
Use Hopital's rule, or heal the function. The limit is 55.

15) T **F** If the total cost $F(x)$ of an entity is extremal at x, then we have a break even point $f(x) = g(x)$.

Solution:
This is not the strawberry theorem.

16) **T** F The value $\int_{-\infty}^{\infty} x f(x) \, dx$ is called the expectation of the PDF f.

Solution:
Yes this is true

17) **T** F The trapezoid rule is an integration method in which the left and right Riemann sum are averaged.

Solution:
This is a good description

18) **T** F $\tan(\pi/3) = \sqrt{3}$.

Solution:
Yes, it is equal to $\sin(\pi/6)$.

19) T **F** A Newton step for the function f is $T(x) = x + \frac{f(x)}{f'(x)}$.

Solution:
Wrong. The sign is off.

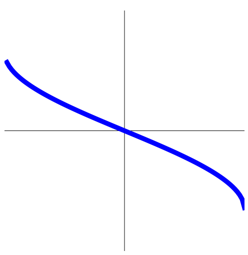
20) T **F** $\sin(\arctan(1)) = \sqrt{3}$.

Solution:
We have $\arctan(1) = \pi/4$ and so $\sin(\arctan(1)) = \sqrt{3}$.

Problem 2) Matching problem (10 points) No justifications needed

(5 points) Match the functions names with their graphs (1-4) their derivatives (A-D) (middle row) and second derivatives (a-d) (last row).

Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$\sin(x)/x$			
$\tan(x)$			
$\arcsin(x)$			
$1/(1+x^2)$			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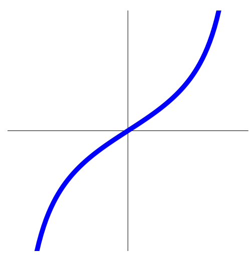
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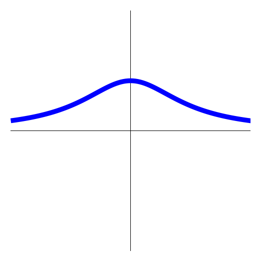
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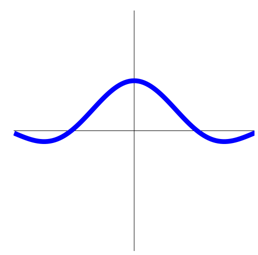
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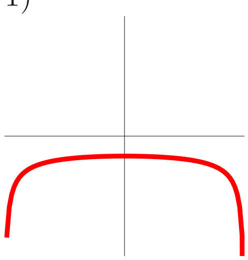
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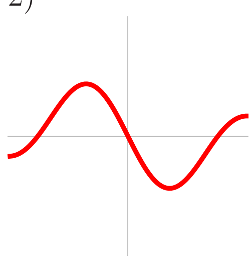
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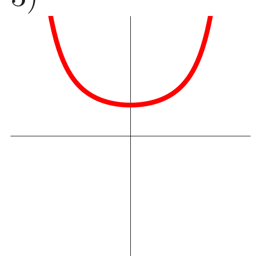
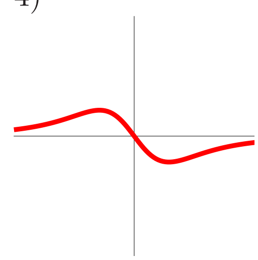

C)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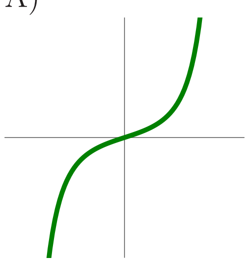
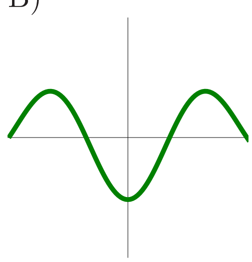

a)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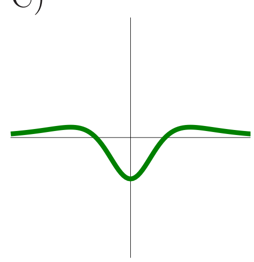
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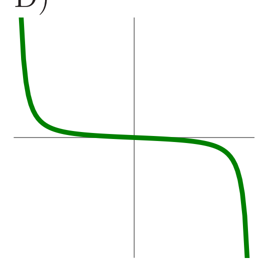

d)

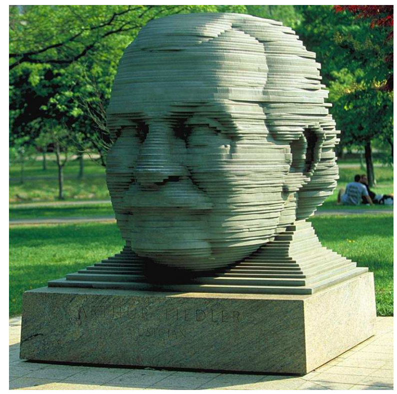
(5 points) Which of the following limits exists in the limit $x \to 0$.

Function	exists	does not exist
$\sin^4(x)/x^4$		
$1/\log\|x\|$		
$\arctan(x)/x$		
$\log\|x\|/(x-1)$		
$\cos(x)/(x-1)$		
$(x^{10}-1)/(x-1)$		

Solution:
a) 4,B,b
2,C,a
1,A,d
3,D,c
b) Every limit exists except the case $\log|x|/(x-1)$.

Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) On the Boston Esplanade is a sculpture of **Arthur Fiedler** (1894-1979) a long-time conductor of the Boston Pops Orchestra. His head is sliced into thin slices. Assume that the thickness of each level is $h = 1.5$ inch and the area of each of the 100 slices k is $A(k)$. Which formula gives the volume of the head? (One applies.)



Formula	Check if true
$1.5[A(1) + \cdots + A(100)]$	
$\frac{1}{1.5}[A(1) + \cdots + A(100)]$	

Formula	Check if true
$1.5[\frac{1}{A(1)} + \cdots + \frac{1}{A(100)}]$	
$\frac{1.5}{100}[A(1) + \cdots + A(100)]$	

b) (4 points) The summer has arrived on May 12 for a day before it cooled down again. Harvard students enjoy the **Lampoon pool** that day in front of the **Lampoon castle**. Assume the water volume at height z is $V(z) = 1 + 5z - \cos(z)$. Assume water evaporates at a rate of $V'(z) = -1$ gallon per day. How fast does the water level drop at $z = \pi/2$ meters? Check the right answer: (one applies)

Rate	Check if true
-6	
$-1/6$	

Rate	Check if true
-4	
$-1/4$	

c) (2 points) Speaking of weather: the temperature on May 13 in Cambridge was 52 degrees Fahrenheit. The day before, on May 12, the temperature had been 85 degrees at some point and had us all dream about **beach time**. Which of the following theorems assures that there was a moment during the night of May 12 to May 13 that the temperature was exactly 70 degrees? (One applies.)

Theorem	check if true
Mean value theorem	
Rolle theorem	

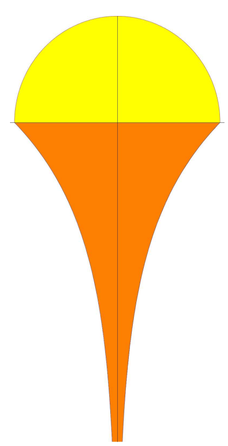
Theorem	check if true
Intermediate value theorem	
Bolzano theorem	

Solution:
$1.5[A(1) + ... + A(100)]$ is the Riemann sum because $dz = 1.5$.
b) $\boxed{z' = -1/6}$ as

$$V' = 5z' + \sin(z)z' = -1$$

gives for $z = \pi/2$ the equation $6z' = -1$ leading to the result. The other selections could made sense if some mistake was done like writing $5 + \sin(z)z' = -1$ for example which would lead to $z' = -6$ which is false.
c) It is the intermediate value theorem.

Problem 4) Area computation (10 points)



Find the area enclosed by the graphs of the functions

$$f(x) = \log|x|$$

and

$$g(x) = \sqrt{1 - x^2}\ .$$

Solution:
The integral is

$$\int_{-1}^{1} \sqrt{1-x^2} - \log|x| \; dx \; .$$

The first integral is $\pi/2$ as it is the area of half the circle. The second integral is $-2\int_0^1 \log|x| \; dx$. $2(x - x\log(x)|_{-1}^{1}) = 4$. The answer is $\boxed{\pi/2 + 4}$.

Problem 5) Volume computation (10 points)

The lamps near the front entrance of the **Harvard Malkin Athletic Center** (MAC) have octagonal cross sections, where at height z, the area is

$$A(z) = 2(1+\sqrt{2})(1+z)^2$$

with $0 \leq z \leq 3$. What is the volume of the lamp?

Solution:
$(2+2\sqrt{2})\int_0^3 (1+z)^2 \; dz = 21(2+\sqrt{8}) = 42 + 21\sqrt{8} = \boxed{42(1+\sqrt{2})}$.

Problem 6) Improper integrals (10 points)

Which of the following limits $R \to \infty$ exist? If the limit exist, compute it.

a) (2 points) $\int_1^R \sin(2\pi x) \; dx$

b) (2 points) $\int_1^R \frac{1}{x^2} \; dx$

c) (2 points) $\int_1^R \frac{1}{\sqrt{x}} \; dx$

d) (2 points) $\int_1^R \frac{1}{1+x^2} \; dx$

e) (2 points) $\int_1^R x \; dx$

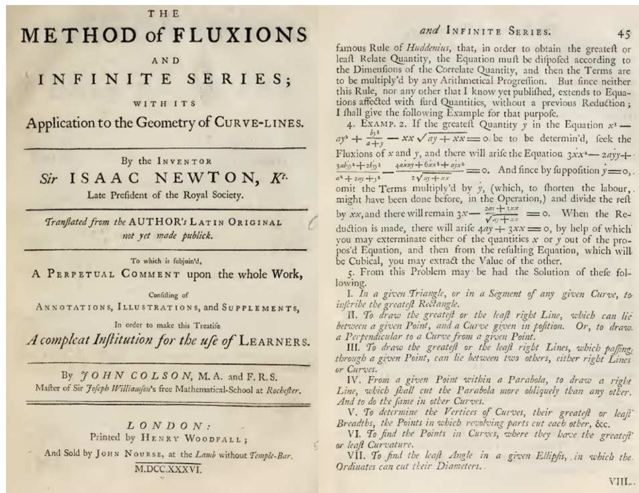
Solution:
a) $\lim_{R\to\infty} -\cos(2\pi x)|_1^R$ does not exist.
b) $\lim_{R\to\infty} -\frac{1}{x}|_1^R = 1$ exists.
c) $\lim_{R\to\infty} 2\sqrt{x}|_1^R$ does not exist.
d) $\lim_{R\to\infty} \arctan(x)_1^R = \pi/2 - \pi/4 = \pi/4$.
e) $\lim_{R\to\infty} R^2/2$ does not exist.

Problem 7) Extrema (10 points)

In Newton's masterpiece **"The Method of Fluxions"** on the bottom of page 45, Newton asks: "In a given triangle or in a segment of any given curve, to inscribe the greatest rectangle." Lets be more specific and find rectangle with largest area

$$A = xy$$

in the triangle given by the x-axes, y-axes and line $y = 2 - 2x$. Use the second derivative test to make sure you have found the maximum.



THE
METHOD of FLUXIONS
AND
INFINITE SERIES;
WITH ITS
Application to the Geometry of CURVE-LINES.

By the INVENTOR
Sir ISAAC NEWTON, Kt.
Late Preſident of the Royal Society.

Tranſlated from the AUTHOR's LATIN ORIGINAL
not yet made publick.

To which is ſubjoin'd,
A PERPETUAL COMMENT upon the whole Work,
Conſiſting of
ANNOTATIONS, ILLUSTRATIONS, and SUPPLEMENTS,
In order to make this Treatiſe
A compleat Inſtitution for the uſe of LEARNERS.

By *JOHN COLSON*, M.A. and F.R.S.
Maſter of Sir *Joſeph Williamſon*'s free Mathematical-School at *Rocheſter*.

LONDON:
Printed by HENRY WOODFALL;
And Sold by JOHN NOURSE, at the *Lamb* without *Temple-Bar*.
M.DCC.XXXVI.

and INFINITE SERIES. 45

famous Rule of *Huddenius*, that, in order to obtain the greateſt or leaſt Relate Quantity, the Equation muſt be diſpoſed according to the Dimenſions of the Correlate Quantity, and then the Terms are to be multiply'd by any Arithmetical Progreſſion. But ſince neither this Rule, nor any other that I know yet publiſhed, extends to Equations affected with ſurd Quantities, without a previous Reduction; I ſhall give the following Example for that purpoſe.

4. EXAMP. 2. If the greateſt Quantity y in the Equation $x^3 - ay^2 + \frac{by^3}{a+y} - xx\sqrt{ay+xx} = 0$ be to be determin'd, ſeek the Fluxions of x and y, and there will ariſe the Equation $3\dot{x}x^2 - 2a\dot{y}y + \frac{3ab\dot{y}y^2 + 2b\dot{y}y^3}{a^2+2ay+y^2} - \frac{4a\dot{x}xy + 6\dot{x}x^3 + a\dot{y}x^2}{2\sqrt{ay+xx}} = 0$. And ſince by ſuppoſition $\dot{y} = 0$, omit the Terms multiply'd by $\dot{y}$, (which, to ſhorten the labour, might have been done before, in the Operation,) and divide the reſt by $\dot{x}x$, and there will remain $3x - \frac{2ay+3xx}{\sqrt{ay+xx}} = 0$. When the Reduction is made, there will ariſe $4ay + 3xx = 0$, by help of which you may exterminate either of the quantities x or y out of the propos'd Equation, and then from the reſulting Equation, which will be Cubical, you may extract the Value of the other.

5. From this Problem may be had the Solution of theſe following.

I. *In a given Triangle, or in a Segment of any given Curve, to inſcribe the greateſt Rectangle.*

II. *To draw the greateſt or the leaſt right Line, which can lie between a given Point, and a Curve given in poſition. Or, to draw a Perpendicular to a Curve from a given Point.*

III. *To draw the greateſt or the leaſt right Lines, which paſſing through a given Point, can lie between two others, either right Lines or Curves.*

IV. *From a given Point within a Parabola, to draw a right Line, which ſhall cut the Parabola more obliquely than any other. And to do the ſame in other Curves.*

V. *To determine the Vertices of Curves, their greateſt or leaſt Breadths, the Points in which revolving parts cut each other, &c.*

VI. *To find the Points in Curves, where they have the greateſt or leaſt Curvature.*

VII. *To find the leaſt Angle in a given Ellipſis, in which the Ordinates can cut their Diameters.*

VIII.

Solution:
The function to extremize is $f(x) = x(2-2x) = 2x - 2x^2$. Its derivative is $f'(x) = 2 - 4x$. It is zero if $\boxed{x = 1/2}$. The second derivative is $f''(x) = -4$. As it is negative, the extremum is a **maximum**.

Problem 8) Integration by parts (10 points)

a) (5 points) Find

$$\int (1 + x + x^2 + x^3 + x^4)(\sin(x) + e^x)\ dx\ .$$

b) (5 points) Find

$$\int \log(x)\frac{1}{x^2}\ dx\ .$$

Solution:
a) Use TicTacToe:

$1+x+x^2+x^3+x^4$	$\sin(x)+e^x$	
$1+2x+3x^2+4x^3$	$-\cos(x)+e^x$	$\oplus$
$2+6x+12x^2$	$-\sin(x)+e^x$	$\ominus$
$6+24x$	$\cos(x)+e^x$	$\oplus$
24	$\sin(x)+e^x$	$\ominus$
0	$-\cos(x)+e^x$	$\oplus$

Collecting together, we could write $(4x^3+3x^2-22x-5)\sin(x)+(x^4-3x^3+10x^2-19x+20)e^x+(-x^4-x^3+11x^2+5x-23)\cos(x)$.
b) As we know from LIATE, we differentiate the log $\log(x)$. We get

$$-\log(x)\frac{1}{x}+\int_x \frac{1}{x^2} = -\log(x)/x - 1/x + C \; .$$

Problem 9) Substitution (10 points)

a) (5 points) **"One,Two,Three,Four Five, once I caught a fish alive!"**

$$\int \frac{(1+2x+3x^2+4x^3+5x^4)}{(1+x+x^2+x^3+x^4+x^5)} \, dx \; .$$

b) (5 points) A **"Trig Trick-or-Treat"** problem:

$$\int (1-x^2)^{-3/2}+(1-x^2)^{-1/2}+(1-x^2)^{1/2} \, dx \; .$$

Solution:
a) Substitute $u = 1+x+x^2+x^3+x^4+x^5$ so that we get $\int du/u = \log(u)+c = \log(1+x+x^2+x^3+x^4+x^5)+C$.
b) Use trig substitution $x=\sin(u)$ in all cases. We get

$$\int \frac{1}{\cos^2(u)} + 1 + \cos^2(u) \, du = \tan(u) + u + (1+\sin(2u)/2)/2 + C$$

which is $\tan(\arcsin(x)) + \arcsin(x) + (1+\sin(2\arcsin(x))/2)/2 + C$.

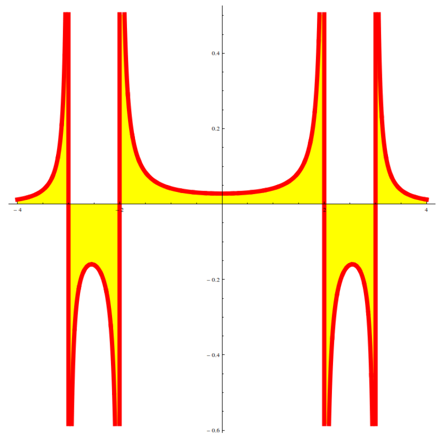
Problem 10) Partial fractions (10 points)

Integrate

$$\int_{-1}^{1} \frac{1}{(x+3)(x+2)(x-2)(x-3)}\, dx \; .$$

The graph of the function is shown to the right.

Lets call it the **friendship graph**.



Solution:
Use partial fraction with the Hopital method of course: Write

$$\frac{1}{(x+3)(x+2)(x-2)(x-3)} = \frac{A}{x+3} + \frac{B}{x+2} + \frac{C}{x-2} + \frac{D}{x-3}$$

To get A, multiply the entire equation with $x+3$, simplify and take the limit $x \to 3$. This gives $A = \frac{1}{(3+3)(3+2)(3-2)} = -1/30$. Similarly, we get $B = \frac{1}{(2+3))(2+2)(2-3)} = -1/20$ and $C = \frac{1}{(-2+3)(-2-2)(-2-3)} = 1/20$ and $D = \frac{1}{(-3+2)(-3-2)(-3-3)} = 1/30$. Now we can write the integral as

$$\begin{aligned}
& \frac{-1}{30} \log|x+3| - \frac{1}{20} \log|x+2| + \frac{1}{20} \log|x-2| + \frac{1}{30} \log|x-3||_{-1}^{1} \\
= & \frac{-1}{30}(\log(4) - \log(2)) - \frac{1}{20} \log(3) + \frac{1}{20} \log(3) + \frac{1}{30}(\log(2) - \log(4)) \\
= & -\frac{2}{30} \log(2) + \frac{2}{20} \log(3) = -\frac{\log(2)}{15} + \frac{\log(3)}{10} \; .
\end{aligned}$$

$\boxed{\log(3)/10 - \log(2)/15}$ is probably the most elegant solution. Results which were not simplified as such were also ok of course. Taking absolute values in the log would not matter neither as logarithms of negative values are technically ok (even so they can be complex, the complex parts cancel).

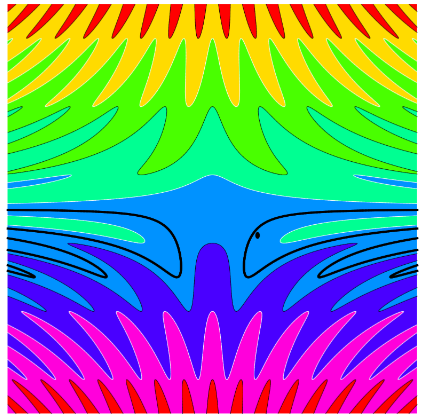
Problem 11) Related rates or implicit differentiation. (10 points)

Assume $x(t)$ and $y(t)$ are related by

$$(\cos(xy) - y) = 1 .$$

We know that $x' = 2$ at $(x, y) = (\pi/2, -1)$. Find y' at this point.

P.S. The figure shows other level curves of a **monster function**. The traced out curve is the curve under consideration.



Solution:
Since a third variable t appears this is a "everybody hates" related rates problem Differentiate with respect to t and solve $-\sin(xy)(x'y + xy') - y' = 0$ with respect to y'. This gives for $x = \pi/2, y = -1, x' = 2$ the answer $-2 + (\pi/2)y' = y'$ so that $y' = 2/(\pi/2 - 1) = \boxed{4/(\pi - 2)}$.

Problem 12) Various integration problems (10 points)

a) (2 points) $\int_0^{2\pi} 2\cos^2(x) - \sin(x)\ dx$

b) (2 points) $\int x^2 e^{3x}\ dx$

c) (2 points) $\int_1^{\infty} \frac{1}{(x+2)^2}\ dx$

d) (2 points) $\int \sqrt{x} \log(x)\ dx$

e) (2 points) $\int_1^e \log(x)^2\ dx$

Solution:
a) Double angle formula: 2π
b) Parts (twice) $e^{3x}(2 - 6x + 9x^2)/27$
c) Substitute $u = x + 2$ to get $1/3$
d) Parts differentiating the log $(3\log(x) - 2)2x^{3/2}/9$
e) parts writing it as $\log(x)^2 \cdot 1$ or as $\log(x) \cdot \log(x)$ (both worked when differentiating the log). The result is $e - 2$

Problem 13) Applications (10 points)

a) (2 points) [**Agnesi density**]

The CDF of the PDF $f(x) = \pi^{-1}/(1+x^2)$ is $\boxed{}$

b) (2 points) [**Piano man**]

The upper hull of $f(x) = x^2 \sin(1000x)$ is the function $\boxed{}$

c) (2 points) [**Rower's wisdom**]

If f is power, F is work and $g = F/x$ then $f = g$ if and only if $g'(x) =$ $\boxed{}$

d) (2 points) [**Catastrophes**]

For $f(x) = c(x-1)^2$ there is a catastrophe at $c =$ $\boxed{}$

e) (2 points) [**Randomness**]

We can use chance to compute integrals. It is called the $\boxed{}$ method.

Solution:
a) Integrate from $-\infty$ to x to get $\arctan(x)/\pi + 1/2$.
b) The function in front is x^2. It gives the amplitude.
c) This is the Strawberry theorem applied to an other situation $g' = 0$.
d) See where the critical point $x = 1$ changes nature: $c = 0$.
e) This was just a knowledge question: Monte Carlo.