

## Lecture 32: Trig substitutions

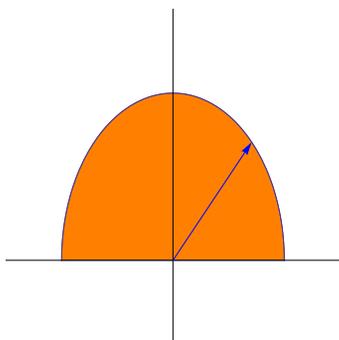
A **Trig substitution** is a special substitution, where  $x$  is a trigonometric function of  $u$  or  $u$  is a trigonometric function of  $x$ . Also this topic is covered more in follow up courses like Math 1b. This lecture allows us to practice more the substitution method. Here is an important example:

- 1 The area of a half circle of radius 1 is given by the integral

$$\int_{-1}^1 \sqrt{1-x^2} dx.$$

**Solution.** Write  $x = \sin(u)$  so that  $\cos(u) = \sqrt{1-x^2}$ .  $dx = \cos(u)du$ . We have  $\sin(-\pi/2) = -1$  and  $\sin(\pi/2) = 1$  the answer is

$$\int_{-\pi/2}^{\pi/2} \cos(u) \cos(u) du = \int_{-\pi/2}^{\pi/2} (1 + \cos(2u))/2 = \frac{\pi}{2}.$$



Lets generalize this a bit and do the same computation for a general radius  $r$ :

- 2 Compute the area of a half disc of radius  $r$  which is given by the integral

$$\int_{-r}^r \sqrt{r^2-x^2} dx.$$

**Solution.** Write  $x = r \sin(u)$  so that  $r \cos(u) = \sqrt{r^2-x^2}$  and  $dx = r \cos(u) du$  and  $r \sin(-\pi/2) = -r$  and  $r \sin(\pi/2) = r$ . The answer is

$$\int_{-\pi/2}^{\pi/2} r^2 \cos^2(u) du = r^2 \pi/2.$$

Here is an example, we know how to integrate

- 3 Find the integral

$$\int \frac{dx}{\sqrt{1-x^2}}.$$

We know the answer is  $\arcsin(x)$ . How can we do that without knowing? **Solution.** We can do it also with a trig substitution. Try  $x = \sin(u)$  to get  $dx = \cos(u) du$  and so

$$\int \frac{\cos(u) du}{\cos(u)} = u = \arcsin(x) + C.$$

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Here is an example, where  $\tan(u)$  is the right substitution. You have to be told that first. It is hard to come up with the idea:

- 4 Find the following integral:

$$\int \frac{dx}{x^2 \sqrt{1+x^2}}$$

by using the substitution  $x = \tan(u)$ . **Solution.** Then  $1+x^2 = 1/\cos^2(u)$  and  $dx = du/\cos^2(u)$ . We get

$$\int \frac{du}{\cos^2(u) \tan^2(u) (1/\cos(u))} = \int \frac{\cos(u)}{\sin^2(u)} du = -1/\sin(u) = -1/\sin(\arctan(x)).$$

Trig substitution is based on the trig identity :

$$\cos^2(u) + \sin^2(u) = 1$$

Depending on whether you divide this by  $\sin^2(u)$  or  $\cos^2(u)$  we get

$$1 + \tan^2(u) = 1/\cos^2(u), 1 + \cot^2(u) = 1/\sin^2(u)$$

These identities are worth remembering. Lets look at more examples:

- 5 Evaluate the following integral

$$\int x^2/\sqrt{1-x^2} dx.$$

**Solution:** Substitute  $x = \cos(u)$ ,  $dx = -\sin(u) du$  and get

$$\int -\frac{\cos^2(u)}{\sin(u)} \sin(u) du = -\int \cos^2(u) du = -\frac{u}{2} - \frac{\sin(2u)}{4} + C = -\frac{\arcsin(x)}{2} + \frac{\sin(2 \arcsin(x))}{4} + C.$$

- 6 Evaluate the integral

$$\int \frac{dx}{(1+x^2)^2}.$$

**Solution:** we make the substitution  $x = \tan(u)$ ,  $dx = du/(\cos^2(u))$ . Since  $1+x^2 = \cos^{-2}(u)$  we have

$$\int \frac{dx}{(1+x^2)^2} = \int \cos^2(u) du = (u/2) + \frac{\sin(2u)}{4} + C = \frac{\arctan(u)}{2} + \frac{\sin(2 \arctan(u))}{4} + C.$$

Here comes an other prototype problem:

- 7 Find the anti derivative of  $1/\sin(x)$ . **Solution:** We use the substitution  $u = \tan(x/2)$  which gives  $x = 2 \arctan(u)$ ,  $dx = 2du/(1+u^2)$ . Because  $1+u^2 = 1/\cos^2(x/2)$  we have

$$\frac{2u}{1+u^2} = 2 \tan(x/2) \cos^2(x/2) = 2 \sin(x/2) \cos(x/2) = \sin(x).$$

Plug this into the integral

$$\int \frac{1}{\sin(x)} dx = \int \frac{1+u^2}{2u} \frac{2du}{1+u^2} = \int \frac{1}{u} du = \log(u) + C = \log(\tan(\frac{x}{2})) + C.$$

Unlike before, where  $x$  is a trig function of  $u$ , now  $u$  is a trig function of  $x$ . This example shows that the substitution  $u = \tan(x/2)$  is magic. Because of the following identities

$$0. u = \tan(x/2)$$

$$\boxed{1} \quad dx = \frac{2du}{1+u^2}$$

$$\boxed{2} \quad \sin(x) = \frac{2u}{1+u^2}$$

$$\boxed{3} \quad \cos(x) = \frac{1-u^2}{1+u^2}$$

It allows us to reduce any rational function involving trig functions to rational functions.

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Any function  $p(x)/q(x)$  where  $p, q$  are trigonometric polynomials can be integrated using elementary functions.

It is usually a lot of work but here is an example:

8 To find the integral

$$\int \frac{\cos(x) + \tan(x)}{\sin(x) + \cot(x)} dx$$

for example, we replace  $dx, \sin(x), \cos(x), \tan(x) = \sin(x)/\cos(x), \cot(x) = \cos(x)/\sin(x)$  with the above formulas we get a rational expression which involves  $u$  only This gives us an integral  $\int p(u)/q(u) du$  with polynomials  $p, q$ . In our case, this would simplify to

$$\int \frac{2u(u^4 + 2u^3 - 2u^2 + 2u + 1)}{(u-1)(u+1)(u^2+1)(u^4 - 4u^2 - 1)} du$$

The method of partial fractions provides us then with the solution.

## Homework

1 Find the antiderivative:

$$\int \sqrt{1-4x^2} dx .$$

2 Find the antiderivative:

$$\int (1-x^2)^{3/2} dx .$$

3 Find the antiderivative:

$$\int \frac{\sqrt{1-x^2}}{x^2} dx .$$

4 Integrate

$$\int \frac{dx}{1+\sin(x)} .$$

Use the substitution  $u = \tan(x/2)$ .

5 Compute

$$\int \frac{dx}{\cos(x)}$$

using the substitution  $u = \tan(x/2)$ .

<sup>1</sup>Proofs:  $\boxed{1}$  differentiate to get  $du = dx/(2\cos^2(x/2)) = dx(1+u^2)/2$ .  $\boxed{2}$  use double angle  $\sin(x) = 2\tan(x/2)\cos^2(x/2)$  and then  $1/\cos^2(x/2) = 1 + \tan^2(x/2)$ .  $\boxed{3}$  use double angle  $\cos(x) = \cos^2(x/2) - \sin^2(x/2) = (1 - \sin^2(x/2))/\cos^2(x/2)$  and again  $1/\cos^2(x/2) = 1 + \tan^2(x/2)$ .