Lecture 19: Fundamental theorem

In this lecture we prove the **fundamental theorem of calculus** for differentiable functions. This will allow us in general to compute integrals of functions which appear as derivatives.

We have seen earlier that with $Sf(x) = h(f(0) + \cdots + f(kh))$ and $Df(x) = (f(x+h) - f(x))/h$ we have $SDf = f(x) - f(0)$ and $DSf(x) = f(x)$ if $x = nh$. This becomes now:

> **Fundamental theorem of calculus:** Assume f is differentiable. Then
>
> $\int_0^x f'(t)\, dt = f(x) - f(0)$ and $\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$

Proof. Using notation of Euler we write $A \sim B$ for "A and B are close" meaning $A - B \to 0$ for $h \to 0$. From $DSf(x) = f(x)$ for $x = kh$ we have $DSf(x) \sim f(x)$ for $kh < x < (k+1)h$ because f is continuous. We also know $\int_0^x Df(t)\, dt \sim \int_0^x f'(t)\, dt$ because $Df(t) \sim f'(t)$ uniformly for $0 \leq t \leq x$ by the definition of the derivative and the assumption that f' is continuous. We also know $SDf(x) = f(x) - f(0)$ for $x = kh$. By definition of the Riemann integral $Sf(x) \sim \int_0^x f(t)\, dt$ and so $SDf(x) \sim \int_0^x Df(t)\, dt$.

$$f(x) - f(0) \sim SDf(x) \sim \int_0^x Df(t)\, dt \sim \int_0^x f'(t)\, dt$$

as well as

$$f(x) \sim DSf(x) \sim D \int_0^x f(t)\, dt \sim \frac{d}{dx} \int_0^x f(t)\, dt .$$

1 $\int_0^5 3t^7\, dt = \frac{x^8}{8}|_0^5 = \frac{5^8}{8}$. You can always leave such expressions as your final result. It is even more elegant than the actual number 390625/8.

2 $\int_0^{\pi/2} \cos(t)\, dt = \sin(x)|_0^{\pi/2} = 1$. This is an important example which should become routine in a while.

3 $\int_0^x \sqrt{1+t}\, dt = \int_0^x (1+t)^{1/2}\, dt = (1+t)^{3/2}(2/3)|_0^x = [(1+x)^{3/2} - 1](2/3)$. Here the difficulty was to see that the $1+t$ in the interior of the function does not make a big difference. Keep such examples in mind.

4 Also in this example $\int_0^2 \cos(t+1)\, dt = \sin(x+1)|_0^2 = \sin(3) - \sin(1)$ the additional term $+1$ does not make a big dent.

5 $\int_{\pi/6}^{\pi/4} \cot(x)\, dx$. This is an example where the anti derivative is difficult to spot. It is easy if we know where to look for: the function $\log(\sin(x))$ has the derivative $\cos(x)/\sin(x)$. So, we know the answer is $\log(\sin(x))|_{\pi/6}^{\pi/4} = \log(\sin(\pi/4)) - \log(\sin(\pi/6)) = \log(1/\sqrt{2}) - \log(1/2) = -\log(2)/2 + \log(2) = \log(2)/2$.

6 The example $\int_1^2 1/(t^2-9)\, dt$ is a bit challenging. We need a hint and write $-6/(x^2-9) = 1/(x+3) - 1/(x-3)$. The function $f(x) = \log|x+3| - \log|x-3|$ has therefore $-6/(x^2-9)$ as a derivative. We know therefore $\int_1^2 -6/(t^2-9)\, dt = \log|3+x| - \log|3-x||_1^2 = \log(5) - \log(1) - \log(4) + \log(2) = \log(5/2)$. The original task is now $(-1/6)\log(5/2)$.

7 $\int_0^x \cos(\sin(x))\cos(x)\, dx = \sin(\sin(x))$ because the derivative of $\sin(\sin(x))$ is $\cos(\sin(x))\cos(x)$. The function $\sin(\sin(x))$ is called the **antiderivative** of f. If we differentiate this function, we get $\cos(\sin(x))\cos(x)$.

8 Find $\int_0^\pi \sin(x)\, dx$. **Solution**: This has a very nice answer.

Here is an important notation, which we have used in the example and which might at first look silly. But it is a handy intermediate step when doing the computation.

> $F|_a^b = F(b) - F(a)$.

We give reformulations of the fundamental theorem in ways in which it is mostly used:

> If f is the derivative of a function F then
>
> $$\int_a^b f(x)\, dx = F(x)|_a^b = F(b) - F(a) .$$

In some textbooks, this is called the "second fundamental theorem" or the "evaluation part" of the fundamental theorem of calculus. The statement $\frac{d}{dx} \int_0^x f(t)\, dt = f(x)$ is the "antiderivative part" of the fundamental theorem. They obviously belong together and are two different sides of the same coin.

Here is a version of the fundamental theorem, where the boundaries are functions of x.

> Given functions g, h and if F is a function such that $F' = f$, then
>
> $$\int_{h(x)}^{g(x)} f(t)\, dt = F(g(x)) - F(h(x)) .$$

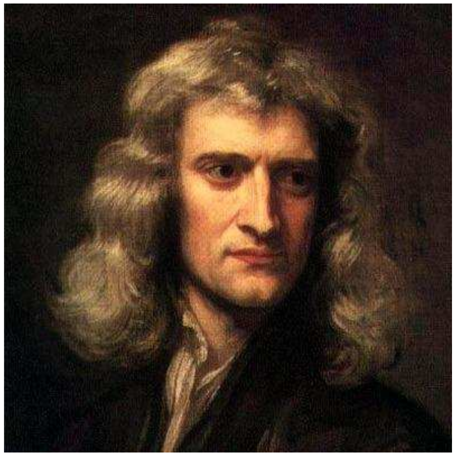
9 $\int_{x^4}^{x^2} \cos(t)\, dt = \sin(x^2) - \sin(x^4)$.

The function F is called an antiderivative. It is not unique but the above formula does always give the right result.

Lets look at a list of important antiderivatives. You should have as many antideriatives "hard wired" in your brain. It really helps. Here are the core functions you should know. They appear a lot.

function	anti derivative
x^n	$\frac{x^{n+1}}{n+1}$
$\sqrt{x}$	$\frac{x^{3/2}2}{3}$
e^{ax}	$\frac{e^{ax}}{a}$
$\cos(ax)$	$\frac{\sin(ax)}{a}$
$\sin(ax)$	$\frac{-\cos(ax)}{a}$
$\frac{1}{x}$	$\log(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\log(x)$	$x\log(x) - x$

Make your own table!



Meet **Isaac Newton** and **Gottfried Leibniz**. They have discovered the fundamental theorem of calculus. You can see from the expression of their faces, they are honored to find themselves on the same handout with **Austin Powers** and **Doctor Evil**.

Homework

1 For any of the following integrals $\int f$, find a function F such that $F' = f$, then integrate a) $\int_0^1 e^{2x} + \sin(3x) + x^3 + 5x \; dx$.
b) $\int_0^1 6(x+4)^3 \; dx$.
c) $\int_2^3 3/x + 4/(x-1) \; dx$.
d) $\int_0^{\sqrt{\pi}} \cos(x^2)x + \sin(x^2)x \; dx$

2 Find the following integrals by finding a function g satisfying $g' = f$. We will learn techniques to find the function. Here, we just use our knowledge about derivatives:
a) $\int_2^3 5x^4 + 4x^3 \; dx$.
b) $\int_{\pi/4}^{\pi/2} \sin(3x) + \cos(x) \; dx$.
c) $\int_{\pi/4}^{\pi/2} \frac{1}{\sin^2(x)} \; dx$.
d) $\int_2^3 \frac{1}{x-1} \; dx$.

3 Evaluate the following integrals:
a) $\int_1^2 2^x \; dx$.
b) $\int_{-1}^1 \cosh(x) \; dx$. (Remember $\cosh(x) = (e^x + e^{-x})/2$.)
c) $\int_0^1 \frac{1}{1+x^2} \; dx$.
d) $\int_{1/3}^{2/3} \frac{1}{\sqrt{1-x^2}} \; dx$.

4 a) Compute $F(x) = \int_0^{x^3} \sin(t) \; dt$, then find $F'(x)$.
b) Compute $G(x) = \int_{\sin(x)}^{\cos(x)} \exp(t) \; dt$ then find $G'(x)$

5 a) **A clever integral:** Evaluate the following integral:
$\int_0^{2\pi} \sin(\sin(\sin(x))) \; dx$
Explain the answer you get.

b) **An evil integral**: Evaluate $\int_e^{e^e} \frac{1}{\log(x)x} \; dx$.
We do not have the tools yet to find this systematically.
Can you figure out a function $f(x)$ which has $1/(\log(x)x)$ as the derivative?