3/1/2012: First hourly Practice A

Your Name:

- Start by writing your name in the above box.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for multiple choice problems, give computations. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.
- All unspecified functions are assumed to be smooth: one can differentiate arbitrarily.
- The actual exam has a similar format: TF questions, multiple choice and then problems where work needs to be shown.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points) No justifications are needed.

1) T **F** The function $\cot(x)$ is the inverse of the function $\tan(x)$.

Solution:
No, it is $\text{artan}(x)$ which is the inverse.

2) **T** F We have $\cos(x)/\sin(x) = \cot(x)$

Solution:
That is the definition.

3) **T** F $\sin(3\pi/2) = -1$.

Solution:
Draw the circle. The angle $3\pi/2$ corresponds to 270 degrees.

4) **T** F The function $f(x) = \sin(x)/x$ has a limit at $x = 0$.

Solution:
Yes, it is called the sinc function.

5) **T** F For the function $f(x) = \sin(\sin(\exp(x)))$ the limit $\lim_{h\to 0}[f(x+h)-f(x)]/h$ exists.

Solution:
The function is differentiable. The limit is the derivative.

6) **T** F If a differentiable function $f(x)$ satisfies $f'(3) = 3$ and is f' is odd then it has a critical point.

Solution:
We have $f'(3) = 3$ and $f'(-3) = -3$. The intermediate value theorem assures that $f'(x) = 0$ for some $x \in [-3, 3]$.

7) T **F** The l'Hopital rule assures that the derivative satisfies $(f/g)' = f'/g'$.

Solution:
The l'Hopital rule tells something about a limit $f(x)/g(x)$ as $x \to p$ but does not compute derivatives.

8) | T | **F** | The intermediate value theorem assures that a continuous function has a derivative.

Solution:
The intermediate value theorem deals with continuous functions.

9) | T | **F** | After healing, the function $f(x) = (x+1)/(x^2-1)$ is continuous everywhere.

Solution:
We divide by zero at $z = 1$.

10) | **T** | F | If f is concave up on $[1, 2]$ and concave down on $[2, 3]$ then 2 is an inflection point.

Solution:
Indeed, f'' changes sign there.

11) | **T** | F | There is a function f which has the property that its second derivative f'' is equal to its negative f.

Solution:
It is the sin or cos function.

12) | **T** | F | The function $f(x) = [x]^4 = x(x-h)(x-2h)(x-3h)$ has the property that $Df(x) = 4[x]^3 = 4x(x-h)(x-2h)$, where $Df(x) = [f(x+h) - f(x)]/h$.

Solution:
Yes, this is a cool property of the polynomials $[x]^n$.

13) | **T** | F | The quotient rule is $d/dx(f/g) = (f'g - fg')/g^2$ and holds whenever $g(x) \neq 0$.

Solution:
This is an important rule to know.

14) T **F** The chain rule assures that $d/dx f(g(x)) = f'(g(x)) + f(g'(x))$.

Solution:
This is not true. We have $f'(g(x))g'(x)$.

15) **T** F If f and g are differentiable, then $(3f + g)' = 3f' + g'$.

Solution:
This is called linearity of the differentiation.

16) T **F** For any function f, the Newton step $T(x)$ is continuous.

Solution:
There is trouble at critical points $f'(x) = 0$.

17) **T** F One can rotate a four legged table on an arbitrary surface such that all four legs are on the ground.

Solution:
We have seen this in class

18) **T** F The fundamental theorem of calculus relates integration S with differentiation D. The result is $DSf(x) = f(x), SDf(x) = f(x) - f(0)$.

19) **T** F The product rule implies $d/dx(f(x)g(x)h(x)) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

Solution:
This was checked in a homework.

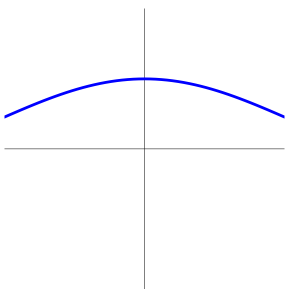
20) T **F** Euler and Gauss are the founders of infinitesimal calculus.

Solution:
It was Newton and Leibniz who are considered the founders, Euler and Gauss came later

Problem 2) Matching problem (10 points) No justifications are needed.

Match the following functions with their graphs.

Function	Fill in 1-8
$x^2 - x$	
$\exp(-x)$	
$\sin(3x)$	
$\log(\|x\|)$	
$\tan(x)$	
$1/(2+\cos(x))$	
$x - \cos(6x)$	
$\sin(3x)/x$	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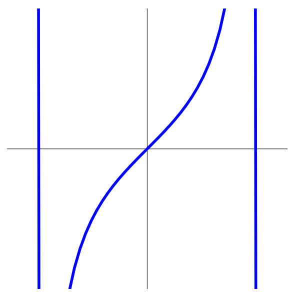
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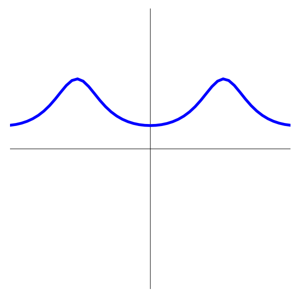
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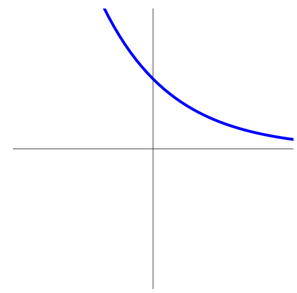
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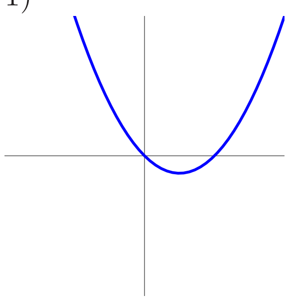
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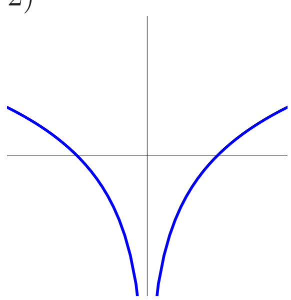
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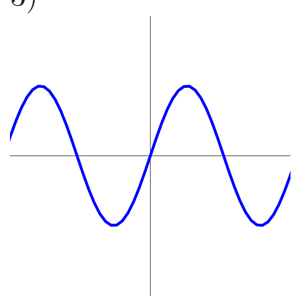
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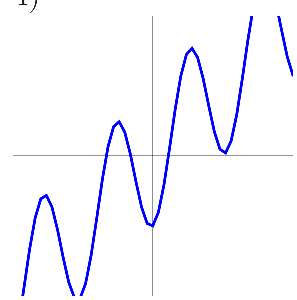

8)

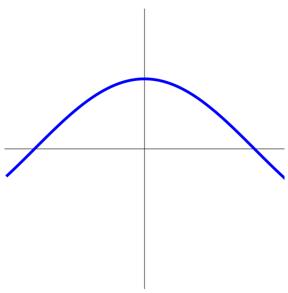
Solution:

Function	Fill in 1-8
$x^2 - x$	5
$\exp(-x)$	4
$\sin(3x)$	7
$\log(\|x\|)$	6
$\tan(x)$	2
$1/(2+\cos(x))$	3
$x - \cos(6x)$	8
$\sin(3x)/x$	1

Problem 3) Matching problem (10 points) No justifications are needed.

Match the following functions with their derivatives.

Function	Fill in the numbers 1-8
graph a)	
graph b)	
graph c)	
graph d)	
graph e)	
graph f)	
graph g)	
graph h)	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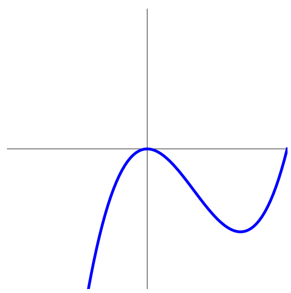
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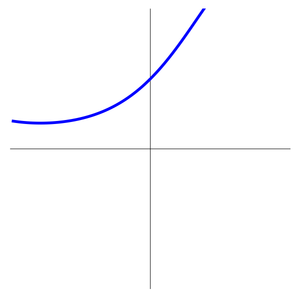
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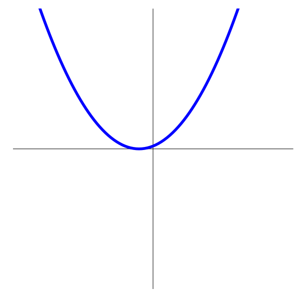
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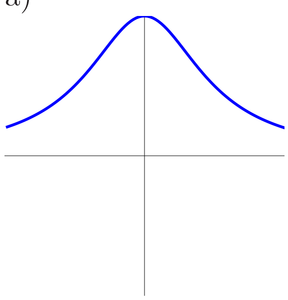
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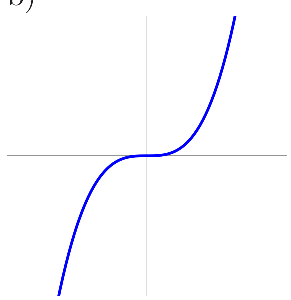
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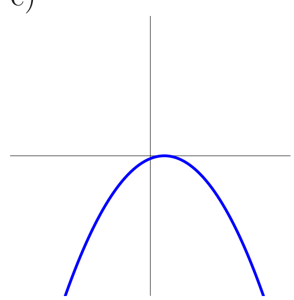
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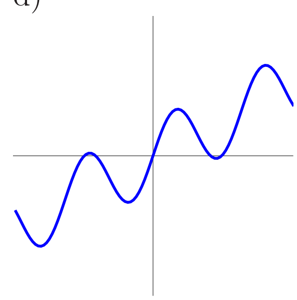
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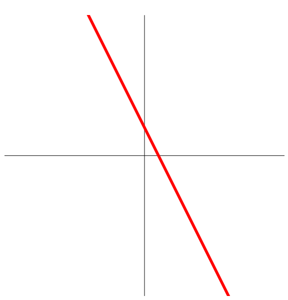
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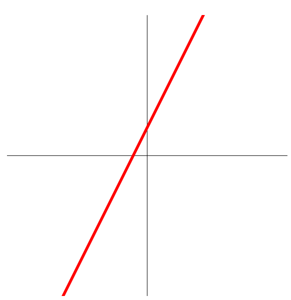
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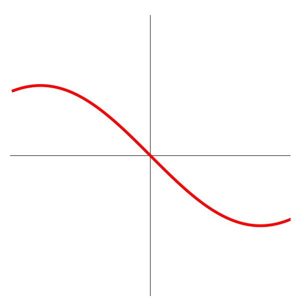
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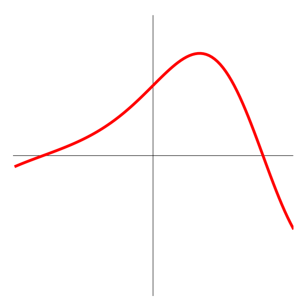
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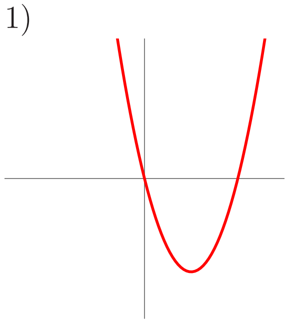
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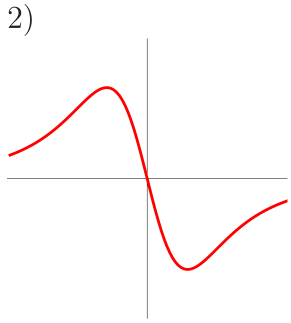
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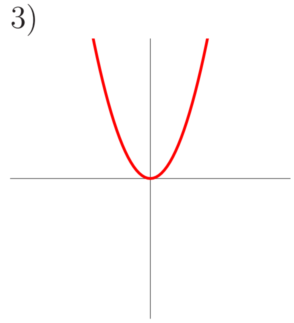
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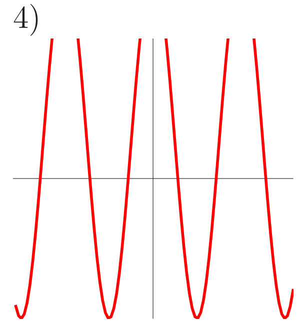

8)

Solution:

Function	Fill in the numbers 1-8
graph a)	3
graph b)	5
graph c)	4
graph d)	2
graph e)	6
graph f)	7
graph g)	1
graph h)	8

Problem 4) Functions (10 points) No justifications are needed

Match the following functions with simplified versions. In each of the rows, exactly one of the choices A-C is true.

Function	Choice A	Choice B	Choice C	Enter A-C
$\frac{x^4-1}{x-1}$	$1+x+x^2+x^3$	$1+x+x^2$	$1+x+x^2+x^3+x^4$	
2^x	$e^{2\log(x)}$	$e^{x\log(2)}$	$2^{e\log(x)}$	
$\sin(2x)$	$2\sin(x)\cos(x)$	$\cos^2(x)-\sin^2(x)$	$2\sin(x)$	
$(1/x)+(1/(2x))$	$1/(x+2x)$	$3/(2x)$	$1/(x+2x)$	
e^{x+2}	$e^x e^2$	$2e^x$	$(e^x)^2$	
$\log(4x)$	$4\log(x)$	$\log(4)\log(x)$	$\log(x)+\log(4)$	
$\sqrt{x}^3$	$x^{3/2}$	$x^{2/3}$	$3\sqrt{x}$	

Solution:
A,B,A,B,A,C,A

Problem 5) Roots (10 points)

Find the roots of the following functions

a) (2 points) $7\sin(3\pi x)$

b) (2 points) $x^5 - x$.

c) (2 points) $\log|ex|$.

d) (2 points) $e^{5x} - 1$

e) (2 points) $8x/(x^2+4) - x$.

Solution:
a) The function is zero if $3x$ is an integer. The solutions are $\boxed{n/3}$, where n is an integer.
b) The function is zero if x is zero or if $x^4 = 1$. The later has the solutions $x = 1, -1$. The roots are $\boxed{0, 1, -1}$.
c) $\log(x) = 0$ if $x - 1$. Therefore, the root is $\boxed{x = 1/e}$ or $\boxed{x = -1/e}$. You might have been tempted to try $x = 1$ which gives $\log(e) = 1$.
d) $e^{5x} = 1$ for $\boxed{x = 0}$.
e) One solution is $x = 0$. We can factor x out. We need to solve then $1 = 8/(x^2+4)$ which means $x^2 + 4 = 8$ or $x^2 = 4$. We have two more solutions, in total $\boxed{x = 0, 2, x = -2}$.

Problem 6) Derivatives (10 points)

Find the derivatives of the following functions:

a) (2 points) $f(x) = \cos(3x)/\cos(10x)$

b) (2 points) $f(x) = \sin^2(x)\log(1+x^2)$

c) (2 points) $f(x) = 5x^4 - 1/(x^2+1)$

d) (2 points) $f(x) = \tan(x) + 2^x$

e) (2 points) $f(x) = \arccos(x)$

Solution:
a) Use the quotient rule

$$\frac{-3\sin(3x)\cos(10x) + 10\sin(10x)\cos(3x)}{\cos^2(10x)} .$$

b) Use the product rule

$$2\sin(x)\cos(x)\log(1+x^2) + \sin^2(x)2x/(1+x^2) .$$

c) $20x^3 + 2x/(x^2+1)^2$
d) $1/\cos^2(x) + e^{x\log(2)}\log(2)$.
e) Use the chain rule on $\cos(\arccos(x)) = x$. This is $-\sin(\arccos(x))\arccos'(x) = 1$ Since $\sin(x) = \sqrt{1-\cos^2(x)}$ we get the derivative $-1/\sqrt{1-x^2}$.

Problem 7) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ of the following functions:

a) (2 points) $f(x) = (x^6 - 3x^2 + 2x)/(1 + x^2 - \cos(x))$.

b) (2 points) $f(x) = (\cos(3x) - 1)/(\cos(7x) - 1)$.

c) (2 points) $f(x) = \tan^3(x)/x^3$.

d) (2 points) $f(x) = \sin(x)\log(x^6)$

e) (2 points) $f(x) = 4x(1-x)/(\cos(x) - 1)$.

Solution:
a) The limit does $\boxed{\text{not exist}}$. After applying l'Hopital once we get a denominator which is zero and a nominator which is nonzero.
b) Use l'Hopital twice. After applying it once, we get $(-3\sin(3x))/(-7\sin(7x))$. Applying l'Hopital again gives $\boxed{9/49}$.
c) First compute the limit $\tan(x)/x$ which is 1 by l'Hopital. The expression is 1^3 which is $\boxed{1}$.
d) We can write this as $\sin(x)6\log(x)$. We can either apply l'Hopital to $6\log(x)/(1/\sin(x))$ and get $6\sin^2(x)/(x\cos(x))$. Since $\sin(x)/x$ goes to 1 we get the limit $\boxed{0}$.
e) like in a), the limit $\boxed{\text{does not exist}}$.

Problem 8) Extrema (10 points)

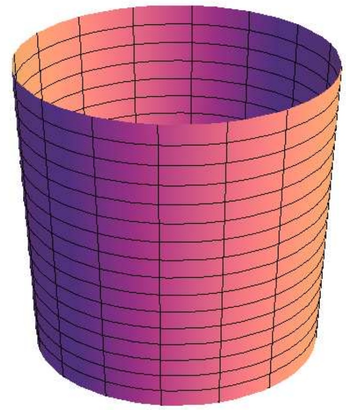
a) (5 points) Find all local extrema of the function $f(x) = 30x^2 - 5x^3 - 15x^4 + 3x^5$ on the real line.

b) (5 points) Find the global maximum and global minimum of the function $f(x) = \exp(x) - \exp(2x)$ on the interval $[-2, 2]$.

Solution:
a) The function $f'(x) = 15x^4 - 60x^3 - 15x^2 + 60x$ has roots at $-1, 0, 1, 4$. We can find them by trying with integers. These are candidates for local extrema. WE can compute the second derivative at these 4 points to get $-150, 60, -90, 900$. The points $\boxed{1, -1 \text{ are local maxima}}$, the points $\boxed{0, 4 \text{ are local minima}}$.
b) The function can be written as $\exp(x)(1 - \exp(2x))$ which has the only root $x = -\log(2)$. In order to find the maxima or minima, we also have to look at the boundary points. The function satisfies $f(-\log(2)) = 1/4, f(2) = e^2 - e^4, f(-2) = e^{-2} - e^{-4}$. $\boxed{\text{The minimum is } -2, \text{ the maximum is } f(-\log(2)) = 1/4.}$.

Problem 9) Extrema (10 points)

A cup of height h and radius r has the volume $V = \pi r^2 h$. Its surface area is $\pi r^2 + \pi r h$. Among all cups with volume $V = \pi$ find the one which has minimal surface area. Find the global minimum.



Solution:
We need to find the minimum of the function $f(r) = \pi(r^2 + 1/r)$. Compute the derivative: $f'(x) = \pi 2r - \pi/r^2$. If this is zero then $2r^3 = 1$ and $\boxed{r = 2^{-1/3}}$. For $r \to 0$ the function goes to infinity as it does for $r \to \infty$. Therefore, the function has a global minimum at $1/2^{1/3}$.

Problem 10) Newton method (10 points)

a) (3 points) Produce the first Newton step for the function $f(x) = e^x - x$ at the point $x = 1$.

b) (4 points) Produce a second Newton step.

c) (3 points) Find the Newton step map $T(x)$ if the function $f(x)$ is replaced by the function $3f(x)$.

Solution:
a) We have $f'(x) = e^x - 1$. The Newton map is $T(x) = x - (e^x - x)/(e^x - 1)$. Applying this map at $x = 1$ gives $1 - (e - 1)/(e - 1) = \boxed{0}$.
b) $T(0) = 0 - 1/0$ pulls us to $\boxed{\text{infinity}}$. The step can not be done. This always happens if we apply the Newton map at a local extremum.
c) The map T does $\boxed{\text{not change}}$ because $f(x)/f'(x)$ does not change if we replace f with $3f$.