

Lecture 37: Final remarks

The holographic picture

How can one summarize calculus in one sentence?

"Calculus establishes that two operations on functions f are related: first the derivative of f which is the rate of change of f second the integral of f which is the area under the graph of f ."

In one paragraph:

The development of calculus by Newton and Leibniz is a major achievement of the past millennium. The core of this course introduces differential and integral calculus. Differential calculus studies the "rate of change" f' of a function, integral calculus treats "accumulation" $\int_0^x f(x) dx$ which can be interpreted as "area under the curve". The fundamental theorem of calculus links the two: it tells that

$$\int_0^x f'(t) dt = f(x) - f(0), \quad \frac{d}{dx} \int_0^x f(t) dt = f(x).$$

The subject can be applied to problems from other scientific disciplines like economics (the strawberry theorem for total, average and marginal costs), reasoning for relating quantities (like estimating the speed of an airplane from angle change, psychology (catastrophes explaining revolutionary changes or flips in human perception), geometry (volume and area computation), statistics (distribution and cumulative distribution functions) or everyday life (the wobbly table theorem).

In one page?

1. Calculus relates two fundamental operations, the derivative measuring the rate of change of a function and the integral measuring the area under the graph.
2. Taking derivatives is done with the chain, product and quotient rules, taking intervals with substitution including trig substitution, integration by parts and partial fraction rules.
3. Basic functions are polynomials and exp, log, sin, cos, tan. We can add, subtract, multiply, divide and compose functions, we can differentiate and integrate them.
4. A function is continuous at p if $f(x) \rightarrow f(p)$ for $x \rightarrow p$. It is differentiable at p if $(f(x+h) - f(x))/h$ has a limit for $h \rightarrow 0$. Limits from the right and left should agree.
5. To extremize a function, we look at points where $f'(x) = 0$. If $f''(x) > 0$ we have a local minimum, if $f''(x) < 0$ we have a local maximum. There can be critical points $f'(x) = 0$ without being extremum like x^3 .
- 6) To relate how different quantities change in time, we differentiate the formula relating the quantities using the chain rule. If there is a third time variable, then this is the story of related rates, if one of the variables is the parameter, then this is implicit differentiation.

One could look at math from a historical perspectives or read original things. One could do projects, use more computer algebra systems, practice visualization and visualize things. You have studied maybe 300 hours for this course including homework, reading, and discussing the

material. Years would be needed to study it more on a research level. New calculus is constantly developed. I myself have been working mostly on more probabilistic versions of calculus which allows to bypass some of the difficulties when discretizing calculus. The loss of symmetries obtained by discretization can be compensated differently.

The future of calculus

Calculus will without doubt look different in 50 years. Many changes have already started, not only on the context level, also from outside: Calculus books will be gone, electronic paper which will be almost indistinguishable from real paper has replaced it. Text, computations, graphics are all fluid in that we can at any point adjust the amount of details. Similarly than we can zoom into a map or picture by pinching the screen, we can triple pinch a text or proof or picture. As we do so, more details are added, more steps of a calculation added, more information included into a graph etc. Every picture is interactive can turn in a movie, an animation, parameters can be changed, functions deformed with the finger. Every picture is a little laboratory. Questions can be asked directly to the text and answers provided. The text can at any time be set back to an official textbook version of the course. The teacher has the possibility to set global preferences and toss around topics. Examples, homework problems and exam problems will be adjusted automatically disallowing for example to treat integration by parts before the product rule. Much of this is not science fiction, there are electronic interactive books already now available for tablet computers which have impressive experimental and animation features. Impossible because it is too difficult to achieve? Remember the last lecture 36. We will have AI on our side and much of this grunt work to compress and expand knowledge can be done computer assisted.

Calculus courses after 1a

To prepare for this course, I set myself the task to formulate the main topic in one short sentence and then single out 4 major goals for the course, then build titles for each lecture etc Here are 4 calculus courses at Harvard drawn out at the level of a "4 point summary". At other schools of higher education, there are similar courses.

The course 1A	from extremization to the fundamental theorem
functions	polynomials, exp, log, trig functions
limits	velocity, tangents, infinite limits
derivatives	product, chain rule with related rates, extremization
integrals	techniques, area, volume, fundamental theorem
The course 1B	from series and integration to differential equations
integration	integration: parts, trig substitution, partial fractions, indefinite
series	convergence, Power, Taylor and Dirichlet series
diff equations	separation of variables, systems like exponential and logistic equations
systems diff eq	equilibria, nullclines, analysis
The course 21A	geometry, extremization and integral theorems in space
geometry	analytic geometry of space, geometric objects, distances
differentiation	curves and surfaces, gradient, curl, divergence
integration	double and triple integrals, other coordinate systems
integral theorems	line and flux integrals, Green, Stokes and Gauss
The course 21B	matrix algebra, eigensystems, dynamical systems and Fourier
equations and maps	Gauss-Jordan elimination, kernel, image, linear maps
matrix algebra	determinants, eigenvalues, eigenspaces, diagonalization
dynamical systems	difference and differential equations with various techniques
fourier theory	Fourier series and dynamical systems on function spaces

There is also a 19a/19b track. The 19a course focuses on models and applications in biology, the 19b course replaces differential equations from 21b with probability theory. The Math 20 course covers linear algebra and multivariable calculus for economists in one semester but covers less material than the 21a/21b track.

The lighter side of calculus

Sofia, our bot had also to know a lot of jokes, especially about math. Here are some relevant to calculus in some way. I left out the inappropriate ones.

- Why do you rarely find mathematicians at the beach? Because they use sine and cosine to get a tan.
- Theorem:** The less you know, the more you make. **Proof:** We know $\text{Power} = \text{Work}/\text{Time}$. Since $\text{Knowledge} = \text{Power}$ and $\text{Time} = \text{Money}$ we know $\text{Knowledge} = \text{Work}/\text{Money}$. Solve for Money to get $\text{Money} = \text{Work}/\text{Knowledge}$. If Knowledge goes to zero, money approaches infinity.
- Why do they never serve beer in a calculus class? Because you can't drink and derive.
- Descartes comes to a bar. Barmen: An other beer? Descartes: I think not. And disappears.
- If it's zero degrees outside today. Tomorrow it will be twice as cold. How cold will it be?
- There are three types of calculus teachers: those who can count and those who can not.
- Calculus is like love; a simple idea, but it can be complicated.
- A mathematician and an engineer are on a desert island with two palm trees and coconuts. The engineer climbs up, gets its coconut gets down and eats. The mathematician climbs up the other, gets the coconut, climbs the first tree and deposits it. "I've reduced the problem to a solved one".
- Pickup line: You are so x^2 . Can I be $x^3/3$, the area under your curves?

- The Evolution of calculus teaching:
1960ies: A peasant sells a bag of potatoes for 10 dollars. His costs are $4/5$ of his selling price. What is his profit?
1970ies: A farmer sells a bag of potatoes for 10 dollars. His costs are $4/5$ of his selling price, that is, 8 dollars. What is his profit?
1980ies: A farmer exchanges a set P of potatoes with a set M of money. The cardinality of the set M is equal to 10, and each element of M is worth one dollars Draw ten big dots representing the elements of M. The set C of production costs is composed of two big dots less than the set M. Represent C as a subset of M and give the answer to the question: What is the cardinality of the set of profits?
1990ies: A farmer sells a bag of potatoes for 10 dollars. His production costs are 8 dollars, and his profit is 2 dollars. Underline the word "potatoes" and discuss it with your classmates.
2000ies: A farmer sells a bag of potatoes for 10 dollars. His or her production costs are 0.80 of his or her revenue. On your calculator, graph revenue vs. costs and run the program POTATO to determine the profit. Discuss the result with other students and start blog about other examples in economics.
2010ies: A farmer sells a bag of potatoes for 10 dollars. His costs are 8 dollars. Use the Potato theorem to find the profit. Then watch the wobbling potato movie.
- Q:** What is the first derivative of a cow? **A:** Prime Rib!
- Q:** What does the zero say to the eight? **A:** Nice belt!
- Theorem.** A cat has nine tails. **Proof.** No cat has eight tails. Since one cat has one more tail than no cat, it must have nine tails.
- Q:** How can you tell that a mathematician is extroverted? **A:** When talking to you, he looks at your shoes instead of at his.
- Q:** What does the little mermaid wear? **A:** An algae-bra.
- In a dark, narrow alley, a function and a differential operator meet: "Get out of my way - or I'll differentiate you till you're zero!" "Try it - I'm e^x ..." Same alley, same function, but a different operator: "Get out of my way - or I'll differentiate you till you're zero!" "Try it - I'm e^x ..." "Too bad... I'm d/dy ."
- Q:** How do you make 1 burn? **A:** Fire differentiation at a log.
- An investment firm hires. In the last round, a mathematician, an engineer, and a business guy are asked what starting salary expectations they had: mathematician: "Would 30,000 be too much?" engineer: "I think 60,000 would be OK." Finance person: "What about 300,000?" Officer: "A mathematician will do the same work for a tenth!" Business guy: "I thought of 135,000 for me, 135,000 for you and 30,000 for the mathematician to do the work.
- Theorem.** Every natural number is interesting. **Proof.** Assume there is an uninteresting one. Then there is smallest one. But as the smallest, it is interesting. Contradiction!