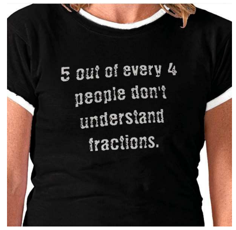
Math 1A: introduction to functions and calculus **Oliver Knill, 2012**

Lecture 31: Partial fractions



The partial fraction method will be covered in detail follow up calculus courses like Math 1b. Here we just look at some samples to see whats out there. We have learned how to integrate polynomials like $x^4 + 5x + 3$. What about rational functions? We will see here that they are a piece of cake - if you have the right guide of course ...

What we know already

Lets see what we know already:

- We also know that integrating $1/x$ gives $\log(x)$. We can for example integrate
$$\int \frac{1}{x-6}\, dx = \log(x-6) + C .$$
- We also have learned how to integrate $1/(1+x^2)$. It was an important integral:
$$\int \frac{1}{1+x^2}\, dx = \arctan(x) + C .$$
Using substitution, we can do more like
$$\int \frac{dx}{1+4x^2} = \int \frac{du/2}{1+u^2} = \arctan(u)/2 = \arctan(2x)/2 .$$
- We also know how to integrate functions of the type $x/(x^2+c)$ using substitution. We can write $u = x^2 + c$ and get $du = 2xdx$ so that
$$\int \frac{x}{x^2+c}\, dx = \int \frac{1}{2u}\, du = \frac{\log(x^2+c)}{2} .$$
- Also functions $1/(x+c)^2$ can be integrated using substitution. With $x + c = u$ we get $du = dx$ and
$$\int \frac{1}{(x+c)^2}\, dx = \int \frac{1}{u^2} du = -\frac{1}{u} + C = -\frac{1}{x+c} + C .$$

The partial fraction method

We would love to be able to integrate any rational function
$$f(x) = \frac{p(x)}{q(x)} ,$$
where p, q are polynomials. This is where **partial fractions come in**. The idea is to write a rational function as a sum of fractions we know how to integrate. The above examples have shown that we can integrate $a/(x+c), (ax+b)/(x^2+c), a/(x+c)^2$ and cases, which after substitution are of this type.

The partial fraction method writes $p(x)/q(x)$ as a sum of functions of the above type which we can integrate.

This is an algebra problem. Here is an important special case:

In order to integrate $\int \frac{1}{(x-a)(x-b)}\, dx$, write
$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} .$$
and solve for A, B.

In order to solve for A, B, write the right hand side as one fraction again
$$\frac{1}{(x-a)(x-b)} = \frac{A(x-b) + B(x-a)}{(x-a)(x-b)} .$$
We only need to look at the nominator:
$$1 = Ax - Ab + Bx - Ba .$$
In order that this is true we must have $A + B = 0, Ab - Ba = 1$. This allows us to solve for A, B.

Examples

1 To integrate $\int \frac{2}{1-x^2}\, dx$ we can write
$$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$
and integrate each term
$$\int \frac{2}{1-x^2} = \log(1+x) - \log(1-x) .$$

2 Integrate $\frac{5-2x}{x^2-5x+6}$. **Solution.** The denominator is factored as $(x-2)(x-3)$. Write
$$\frac{5-2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2} .$$
Now multiply out and solve for A, B:
$$A(x-2) + B(x-3) = 5 - 2x .$$
This gives the equations $A + B = -2, -2A - 3B = 5$. From the first equation we get $A = -B - 2$ and from the second equation we get $2B + 4 - 3B = 5$ so that $B = -1$ and so $A = -1$. We have not obtained
$$\frac{5-2x}{x^2-5x+6} = -\frac{1}{x-3} - \frac{1}{x-2}$$
and can integrate:
$$\int \frac{5-2x}{x^2-5x+6}\, dx = -\log(x-3) - \log(x-2) .$$
Actually, we could have got this one also with substitution. How?

3 Integrate $f(x) = \int \frac{1}{1-4x^2}\, dx$. **Solution.** The denominator is factored as $(1-2x)(1+2x)$. Write
$$\frac{A}{1-2x} + \frac{B}{1+2x} = \frac{1}{1-4x^2} .$$
We get $A = 1/4$ and $B = -1/4$ and get the integral
$$\int f(x)\, dx = \frac{1}{4}\log(1-2x) - \frac{1}{4}\log(1+2x) + C .$$

Hopital's method

There is a fast method to get the coefficients:

If a is different from b, then the coefficients A, B in

$$\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} ,$$

are

$$A = \lim_{x \to a} (x-a)f(x) = p(a)/(a-b), \; B = \lim_{x \to b} (x-b)f(x) = p(b)/(b-a) .$$

Proof. If we multiply the identity with $x - a$ we get

$$\frac{p(x)}{(x-b)} = A + \frac{B(x-a)}{x-b} .$$

Now we can take the limit $x \to a$ without peril and end up with $A = p(a)/(x-b)$.

Cool, isn't it? This **Hopital method** can save you a lot of time! Especially when you deal with more factors and where sometimes complicated systems of linear equations would have to be solved. Remember

Math is all about elegance and does not use complicated methods if simple ones are available.

Here is an example:

4 Find the anti-derivative of $f(x) = \frac{2x+3}{(x-4)(x+8)}$. **Solution.** We write

$$\frac{2x+3}{(x-4)(x+8)} = \frac{A}{x-4} + \frac{B}{x+8}$$

Now $A = \frac{2*4+3}{4+8} = 11/12$, and $B = \frac{2*(-8)+3}{(-8-4)} = 13/12$. We have

$$\frac{2x+3}{(x-4)(x+8)} = \frac{(11/12)}{x-4} + \frac{(13/12)}{x+8} .$$

The integral is

$$\frac{11}{12} \log(x-4) + \frac{13}{12} \log(x+8) .$$

Here is an example with three factors:

5 Find the anti-derivative of $f(x) = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}$. **Solution.** We write

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Now $A = \frac{1^2+1+1}{(1-2)(1-3)} = 3/2$ and $B = \frac{2^2+2+1}{(2-1)(2-3)} = -7$ and $C = \frac{3^2+3+1}{(3-1)(3-2)} = 13/2$. The integral is

$$\frac{3}{2} \log(x-1) - 7\log(x-2) + \frac{13}{2} \log(x-3) .$$

Homework

1 $\int \frac{2dx}{x^2-4}$.

2 $\int \frac{5dx}{4x^2+1}$.

3 $\int \frac{x^3-x+1}{x^2-1} \, dx$.

4 $\int \frac{3x^2}{(x^2+x+1)(x-1)} \, dx$

5 $\int \frac{1}{(x+1)(x-1)(x+7)(x-3)} \, dx$. Use Hopitals method of course!

Hint for 3). Subtract first a polynomial.

Hint for 4). Find the nominator of $\frac{Ax+B}{x^2+x+1} + \frac{C}{x-1}$ and set it $3x^2$. To do so, multply out.