

Lecture 29: Integration by parts

If we integrate the product rule $(uv)' = u'v + uv'$ we obtain an integration rule called **integration by parts**. It is a powerful tool, which complements substitution. As a rule of thumb, always try first to simplify a function and integrate directly, then give substitution a first shot before trying integration by parts.

$$\int u(x) v'(x) dx = u(x)v(x) - \int u'(x)v(x) dx.$$

- 1 Find $\int x \sin(x) dx$. **Solution.** Lets identify the part which we want to differentiate and call it u and the part to integrate and call it v' . The integration by parts method now proceeds by writing down uv and subtracting a new integral which integrates $u'v$:

$$\int x \sin(x) dx = x (-\cos(x)) - \int 1 (-\cos(x)) dx = -x \cos(x) + \sin(x) + C dx.$$

- 2 Find $\int x e^x dx$. **Solution.**

$$\int x \exp(x) dx = x \exp(x) - \int 1 \exp(x) dx = x \exp(x) - \exp(x) + C dx.$$

- 3 Find $\int \log(x) dx$. **Solution.** There is only one function here, but we can look at it as $\log(x) \cdot 1$

$$\int \log(x) \cdot 1 dx = \log(x)x - \int \frac{1}{x} dx = x \log(x) - x + C.$$

- 4 Find $\int x \log(x) dx$. **Solution.** Since we know from the previous problem how to integrate log we could proceed like this. We would get through but what if we do not know? Lets differentiate $\log(x)$ and integrate x :

$$\int \log(x) x dx = \log(x) \frac{x^2}{2} - \int \frac{1}{x} \frac{x^2}{2} dx$$

which is $\log(x)x^2/2 - x^2/4$.

We see that it is better to differentiate log first.

- 5 **Marry go round:** Find $I = \int \sin(x) \exp(x) dx$. **Solution.** Lets integrate $\exp(x)$ and differentiate $\sin(x)$.

$$= \sin(x) \exp(x) - \int \cos(x) \exp(x) dx.$$

Lets do it again:

$$= \sin(x) \exp(x) - \cos(x) \exp(x) - \int \sin(x) \exp(x) dx.$$

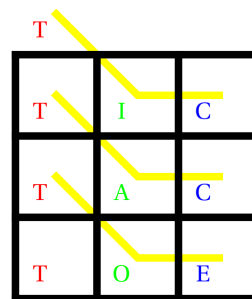
We moved in circles and are stuck! Are we really. We have derived an identity

$$I = \sin(x) \exp(x) - \cos(x) \exp(x) - I$$

which we can solve for I and get

$$I = [\sin(x) \exp(x) - \cos(x) \exp(x)]/2.$$

Tic-Tac-Toe



Integration by parts can bog you down if you do it several times. Keeping the order of the signs can be daunting. This is why a **tabular integration by parts method** is so powerful. It has been called "Tic-Tac-Toe" in the movie Stand and deliver. Lets call it Tic-Tac-Toe therefore.

- 6 Find the anti-derivative of $x^2 \sin(x)$. **Solution:**

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	\oplus
2	$-\sin(x)$	\ominus
0	$\cos(x)$	\oplus

The antiderivative is

$$-x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + C.$$

- 7 Find the anti-derivative of $(x-1)^3 e^{2x}$. **Solution:**

$(x-1)^3$	$\exp(2x)$	
$3(x-1)^2$	$\exp(2x)/2$	\oplus
$6(x-1)$	$\exp(2x)/4$	\ominus
6	$\exp(2x)/8$	\oplus
0	$\exp(2x)/16$	\ominus

The anti-derivative is

$$(x-1)^3 e^{2x}/2 - 3(x-1)^2 e^{2x}/4 + 6(x-1) e^{2x}/8 - 6e^{2x}/16 + C.$$

- 8 Find the anti-derivative of $x^2 \cos(x)$. **Solution:**

x^2	$\cos(x)$	
$2x$	$\sin(x)$	\oplus
2	$-\cos(x)$	\ominus
0	$-\sin(x)$	\oplus

The anti-derivative is

$$x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C.$$

Ok, we are now ready for more extreme stuff.

9 Find the anti-derivative of $x^7 \cos(x)$. **Solution:**

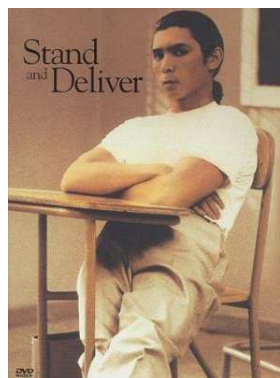
x^7	$\cos(x)$	
$7x^6$	$\sin(x)$	\oplus
$42x^5$	$-\cos(x)$	\ominus
$120x^4$	$-\sin(x)$	\oplus
$840x^3$	$\cos(x)$	\ominus
$2520x^2$	$\sin(x)$	\oplus
$5040x$	$-\cos(x)$	\ominus
5040	$-\sin(x)$	\oplus
0	$\cos(x)$	\ominus

The anti-derivative is

$$\begin{aligned}
 F(x) &= x^7 \sin(x) \\
 &+ 7x^6 \cos(x) \\
 &- 42x^5 \sin(x) \\
 &- 210x^4 \cos(x) \\
 &+ 840x^3 \sin(x) \\
 &+ 2520x^2 \cos(x) \\
 &- 5040x \sin(x) \\
 &- 5040 \cos(x) + C .
 \end{aligned}$$

Do this without this method and you see the value of the method.

1 2 3.



I myself learned the method from the movie "Stand and Deliver", where **Jaime Escalante** of the Garfield High School in LA uses the method. It can be traced down to an article of V.N. Murty. The method realizes in a clever way an iterated integration by parts method:

$$\begin{aligned}
 \int fg dx &= fg^{(-1)} - f^{(1)}g^{-2} + f^{(2)}g^{-3} - \dots \\
 &- (-1)^n \int f^{(n+1)}g^{(-n-1)} dx
 \end{aligned}$$

which can easily shown to be true by induction and justifies the method: the f function is differentiated again and again and the g function is integrated again and again. You see, where the alternating minus signs come from. You see that we always pair a k 'th derivative with a $k + 1$ 'th integral and take the sign $(-1)^k$.

Coffee or Tea?

When doing integration by parts, We want to try first to differentiate **L**ogs, **I**nverse trig functions, **P**owers, **T**rig functions and **E**xponentials. This can be remembered as **LIPTE** which is close to "lipton" (the tea).

For coffee lovers, there is an equivalent one: **L**ogs, **I**nverse trig functions, **A**lgebraic functions, **T**rig functions and **E**xponentials which can be remembered as **LIATE** which is close to "latte" (the coffee).

Whether you prefer to remember it as a "coffee latte" or a "lipton tea" is up to you.

There is even a better method, the "opportunistic method":

Just integrate what you can integrate and differentiate the rest.

An don't forget to consider integrating 1, if nothing else works.



Homework

- 1 Integrate $\int x^2 \log(x) dx$.
- 2 Integrate $\int x^5 \sin(x) dx$
- 3 Find the anti derivative of $\int x^6 \exp(x) dx$. (*)
- 4 Find the anti derivative of $\int \sqrt{x} \log(x) dx$.
- 5 Find the anti derivative of $\int \sin(x) \exp(-x) dx$.

(*) If you want to go for the record. Lets see who can integrate the largest $x^n \exp(x)$! It has to be done by hand, not with a computer algebra system although.



¹V.N. Murty, Integration by parts, Two-Year College Mathematics Journal 11, 1980, pages 90-94.
²David Horowitz, Tabular Integration by Parts, College Mathematics Journal, 21, 1990, pages 307-311.
³K.W. Folley, integration by parts, American Mathematical Monthly 54, 1947, pages 542-543