Lecture 25: Related rates

Before we continue with integration, we include a short flash-back on differentiation. which allows us to review the **chain rule**

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) .$$

This rule will be useful for us for the integration technique called "substitution". Since the chain rule is often perceived as a difficult concept in calculus, it is good to review it again and have fresh breath before launching into more advanced integration techniques. Related rates problem deal with a relation for variables. Differentiation gives a relation between the derivatives (rate of change). In all these problems, we have an **equation** and a **rate**. You can then solve for the rate which is asked for.

1 Hydrophilic **water gel spheres** have volume $\boxed{V(r(t)) = 4\pi r(t)^3/3}$ and expand at a rate $\boxed{V' = 30}$. Find $r'(t)$. **Solution:** $30 = 4\pi r^2 r'$. We get $r' = 30/(4\pi r^2)$.

2 A **wine glass** has a shape $y = x^2$ and volume $\boxed{V(y) = y^2\pi/2}$. Assume we slurp the wine with constant rate $\boxed{V' = -0.1}$. With which speed does the height decrease? We have $d/dtV(y(t)) = V'(y)y'(t) = \pi y y'(t)$ so that $y'(t) = -1/(\pi y)$.

3 A **ladder** has length 1. Assume slips on the ground away with constant speed $\boxed{x' = 2}$. What is the speed of the top part of the ladder sliding down the wall at the time when $x = y$ if $\boxed{x^2(t) + y^2(t) = 1}$. Differentiation gives $2x(t)x'(t) + 2y(t)y'(t) = 0$. We get $y'(t) = -x'(t)x(t)/y(t) = 2 \cdot 1 = 1$.

4 A **kid slides** down a slide of the shape $\boxed{y = 2/x}$. Assume $\boxed{y' = -7}$. What is $x'(t)$? Evaluate it at $x = 1$. **Solution:** differentiate the relation to get $y' = -2x'/x^2$. Now solve for x' to get $x' = -y'x^2/2 = 7/2$.

Image source: http://www.dmfco.com

5 A **canister of oil** releases oil so that the area grows at a constant rate $\boxed{A' = 5}$. With what rate does the radius increase? **Solution.** We have $\boxed{A(r) = r^2\pi}$ and so $5 = A'(r) = 2rr'\pi$. Solving for r' gives $r' = 5/(2r\pi)$.

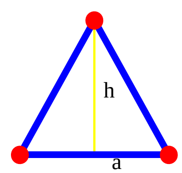
Related rates problems link quantities by a $\boxed{\text{rule}}$. These quantities can depend on time. To solve a related rates problem, differentiate the $\boxed{\text{rule}}$ with respect to time use the given $\boxed{\text{rate of change}}$ and solve for the unknown rate of change. Since related change problems are often difficult to parse. We have the $\boxed{\text{rule}}$ and given $\boxed{\text{rate of change}}$ boxed.

Homework

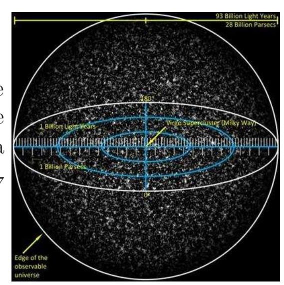
1 The **ideal gas law** $\boxed{pV = T}$ relates pressure p and volume V and temperature T. Assume the temperature $T = 50$ is fixed and $\boxed{V' = -3}$. Find the rate p' with which the pressure increases.

2 Assume the **total production rate** P of a new tablet computer product for kids is constant $P = 100$ and given by the **Cobb-Douglas formula** $\boxed{P = L^{1/3}K^{2/3}}$. Assume labor is increased at a rate $\boxed{L' = 2}$. What is the cost change K'? Evaluate this at $K = 125$ and $L = 64$.

3 You observe an **airplane** at height $h = 10'000$ meters directly above you and see that it moves with rate $\boxed{\phi' = 5}$ degree per second (which is $5\pi/180$ radiants per second). What is the speed x' of the airplane directly above you where $x = 0$? Hint: Use $\boxed{\tan(\phi) = x/h}$.



4 An **isosceles triangle** with base $2a$ and height h has fixed area $\boxed{A = ah = 1}$. Assume the height is decreased by a rate $\boxed{h' = -2}$. With what rate does a increase if $h = 1/2$?



5 There are **cosmological models** which see our universe as a four dimensional sphere which expands in space time. Assume the volume $\boxed{V = \pi^2 r^4/2}$ increases at a rate $\boxed{V' = 100\pi^2 r^2}$. What is r'? Evaluate it for $r = 47$ (billion light years).