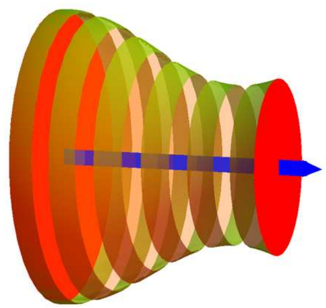
Lecture 22: Volume computation

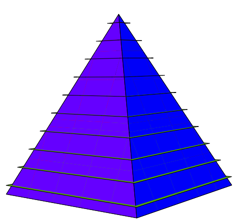
To compute the volume of a solid, we cut it into slices, where each slice is perpendicular to a given line x. If $A(x)$ is the area of the slice and the body is enclosed between a and b then $V = \int_a^b A(x)\, dx$ is the volume. Think of $A(x)dx$ as the volume of a slice. The integral adds them up.



1 Compute the volume of a pyramid with square base length 2 and height 2. **Solution**: we can assume the pyramid is built over the square $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. The cross section area at height h is $A(h) = (2-h)^2$. Therefore,

$$V = \int_0^2 (2-h)^2 \, dh = \frac{8}{3} .$$

This is base area 4 times height 2 divided by 3.



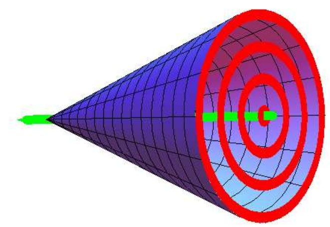
A **solid of revolution** is a surface enclosed by the surface obtained by rotating the graph of a function $f(x)$ around the x-axis.

The area of the cross section at x of a solid of revolution is $A(x) = \pi f(x)^2$. The volume of the solid is $\int_a^b \pi f(x)^2 \, dx$.

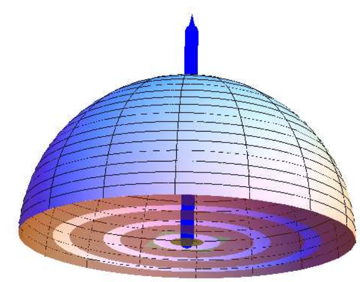
2 Find the volume of a round cone of height 2 and where the circular base has the radius 1. **Solution**. This is a solid of revolution obtained by rotation the graph of $f(x) = x/2$ around the x axes. The area of a cross section is $\pi x^2/4$. Integrating this up from 0 to 2 gives

$$\int_0^2 \pi x^2/4 \, dx = \frac{x^3}{4 \cdot 3}|_0^2 = \frac{2\pi}{3} .$$

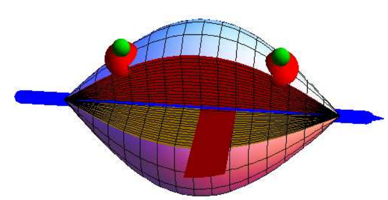
This is the height 2 times the base area π divided by 3.



3 Find the volume of a half sphere of radius 1. **Solution:** The area of the cross section at height h is $\pi(1-h^2)$.



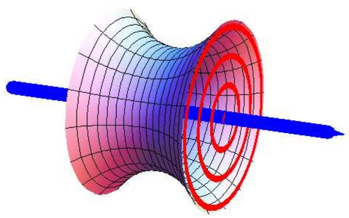
4 We rotate the graph of the function $f(x) = \sin(x)$ around the x axes. But now we cut out a slice of $60 = \pi/3$ degrees out. Find the volume of the solid.
Solution: The area of a slice without the missing piece is $\pi \sin^2(x)$. The integral $\int_0^\pi \sin^2(x)\, dx$ is $\pi/2$ as derived in the lecture. Having cut out 1/6'th the area is $(5/6)\pi \sin^2(x)$. The volume is $\int_0^\pi (5/6)\pi \sin(x)^2 \, dx = (5/6)\pi^2/2$.



Homework

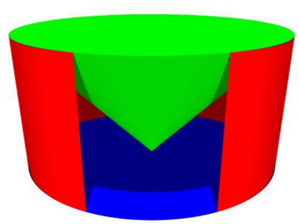
1 Find the volume of the paraboloid for which the radius at position x is $4 - x^2$ and x ranges from 0 to 2.

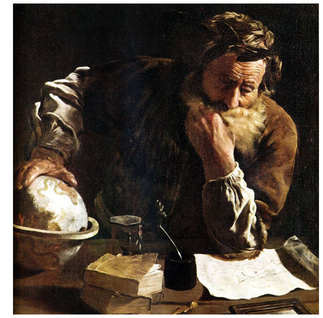
2 A **catenoid** is the surface obtained by rotating the graph of $f(x) = \cosh(x)$ around the x-axes. We have seen that the graph of f is the chain curve, the shape of a hanging chain. Find the volume of of the solid enclosed by the catenoid between $x = -2$ and $x = 2$.
Hint. You might want to check first the identity $2\cosh(x)^2 = 1 + \cosh(2x)$ using the definition $\cosh(x) = (\exp(x) + \exp(-x))/2$.



3 A **tomato** is given by $z^2 + x^2 + 4y^2 = 1$. If we slice perpendicular to the y axes, we get a circular slice $z^2 + x^2 \leq 1 - 4y^2$ of radius $\sqrt{1 - 4y^2}$.
a) Find the area of this slice.
b) Determine the volume of the tomato.
c) Fix yourself a tomato salad by cutting a fresh tomato into slices and eat it, except for one slice which you staple to your homework paper as proof that you really did it.

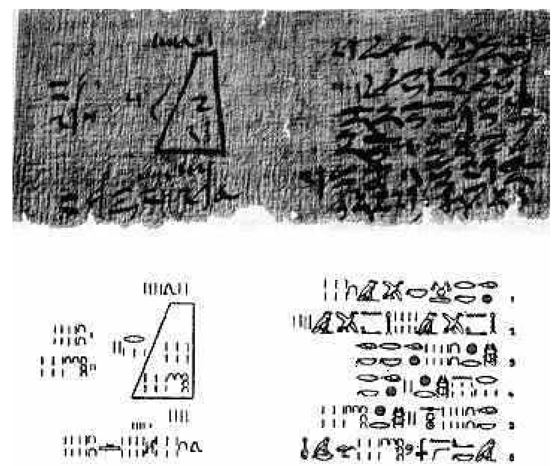
4 As we have seen in the movie of the first class, **Archimedes** was so proud of his formula for the volume of a sphere that he wanted the formula on his tomb stone. He wrote the volume of a half sphere of radius 1 as the difference between the volume of a cylinder of radius 1 and height 1 and the volume of a cone of base radius 1 and height 1. Relate the cross section area of the cylinder-cone complement with the cross section area of the sphere to recover his argument! If stuck, draw in the sand or soak in the bath tub for a while eating your tomato salad. There is no need to streak and scream "Eureka" when the solution is found.





5 Volumes were among the first quantities, Mathematicians wanted to measure and compute. One problem on **Moscow Eqyptian papyrus** dating back to 1850 BC explains the general formula $h(a^2 + ab + b^2)/3$ for a truncated pyramid with base length a, roof length b and height h.
a) Verify that if you slice the frustrum at height z, the area is $(a + (b - a)z/h)^2$.
b) Find the volume using calculus.
Here is the translated formulation from the papyrus: [1] [2]

"You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4 result 16. You are to double 4 result 8. You are to square 2, result 4. You are to add the 16, the 8 and the 4, result 28. You are to take one-third of 6 result 2. You are to take 28 twice, result 56. See it is 56. You will find it right".



[1] Howard Eves, Great moments in mathematics, Volume 1, MAA, Dolciani Mathematical Expositions, 1980, page 10

[2] Image Source: http://www-history.mcs.st-and.ac.uk/HistTopics/Egyptian_papyri.html