Lecture 20: Antiderivatives

We have looked at the integral $\int_0^x f(t)\,dt$ and seen that it is the **signed area under the curve**. The area of the region below the curve is counted in a negative way. There is something else to mention:

For $x < 0$, we define $\int_x^0 f(t)\,dt$ as $-\int_0^x f(t)\,dt$. This is compatible with the fundamental theorem $\int_a^b f'(t)\,dt = f(b) - f(a)$.

The function $g(x) = \int_0^x f(t)\,dt + C$ is called the **anti-derivative** of g. The constant C is arbitrary and not fixed. As we will see below, we can often adjust the constant such that some condition is satisfied.

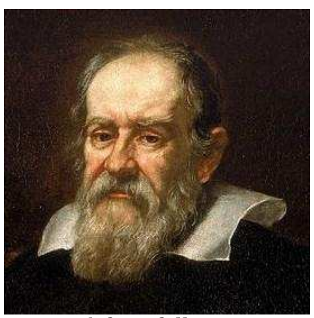
The fundamental theorem of calculus assured us that

The anti derivative is the inverse operation of the derivative. Two different anti derivatives differ by a constant.

Finding the anti-derivative of a function is much harder than finding the derivative. We will learn some techniques but it is in general not possible to give anti derivatives for even very simple functions.

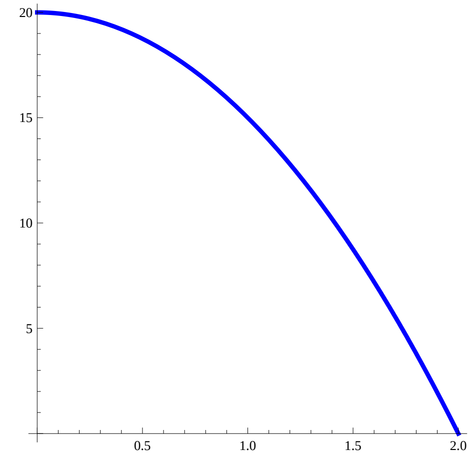
1 Find the anti-derivative of $f(x) = \sin(4x) + 20x^3 + 1/x$. Solution: We can take the anti-derivative of each term separately. The anti derivative is $F(x) = -\cos(4x)/4 + 4x^4 + \log(x) + C$.

2 Find the anti derivative of $f(x) = 1/\cos^2(x) + 1/(1-x)$. **Solution:** we can find the anti derivatives of each term separately and add them up. The result is $F(x) = \cot(x) + \log|1-x| + C$.



Galileo

3 measured **free fall**, a motion with constant acceleration. Assume $s(t)$ is the height of the ball at time t. Assume the ball has zero velocity initially and is located at height $s(0) = 20$. We know that the velocity is $v(t)$ is the derivative of $s(t)$ and the acceleration $a(t)$ is constant equal to -10. So, $v(t) = -10t + C$ is the antiderivative of a. By looking at v at time $t = 0$ we see that $C = v(0)$ is the initial velocity and so zero. We know now $v(t) = -10t$. We need now to compute the anti derivative of $v(t)$. This is $s(t) = -10t^2/2 + C$. Comparing $t = 0$ shows $C = 20$. Now $s(t) = 20 - 5t^2$. The graph of s is a parabola. If we give the ball an additional horizontal velocity, such that time t is equal to x then $s(x) = 20 - 5x^2$ is the visible trajectory. We see that jumping from 20 meters leads to a fall which lasts 2 seconds.



4 The **total cost** is the antiderivative of the **marginal cost** of a good. Both the marginal cost as well as the total cost are a function of the quantity produced. For instance, suppose the total cost of making x shoes is 300 and the total cost of making $x+4$ shoes is 360 for all x. The marginal cost is $60/4 = 15$ dollars. In general the marginal cost changes with the number of goods. There is additional cost needed to produce one more shoe if 300 shoes are produced. **Problem:** Assume the marginal cost of a book is $f(x) = 5 - x/100$ and that producing the first 10 books costs 1000 dollars. What is the total cost of producing 100 books? **Answer:** The anti derivative $5 - x/100$ of f is $F(x) = 5x - x^2/100 + C$ where C is a constant. By comparing $F(10) = 1000$ we get $50 - 100/100 + C = 1000$ and so $C = 951$. the result is $951 + 5 * 100 - 10'000/100 = 1351$. The average book prize has gone down from 100 to 13.51 dollars.

5 The **total revenue** $F(x)$ is the antiderivative of the **marginal revenue** $f(x)$. Also these functions depend on the quantity x produced. We have $F(x) = P(x)x$, where $P(x)$ is the prize. Then $f(x) = F'(x) = P'(x)x + P$. For a **perfect competitive market**, $P'(x) = 0$ so that the prize is equal to the marginal revenue.

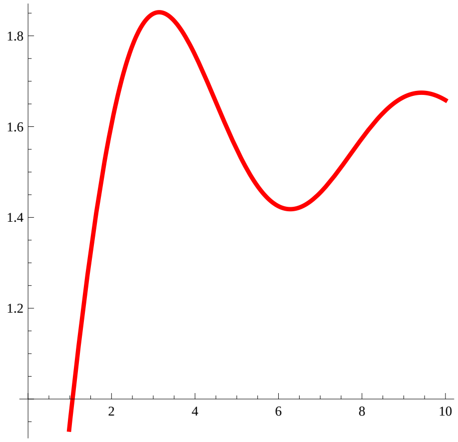
A function f is called **elementary**, if it can be constructed using addition, subtraction, multiplication, division, compositions from polynomials or roots. In other words, an elementary function is built up with functions like $x^3, \sqrt{\cdot}, \exp, \log, \sin, \cos, \tan$ and $\arcsin, \arccos, \arctan$.

6 The function $f(x) = \sin(\sin(\pi + \sqrt{x} + x^2)) + \log(1 + \exp((x^6+1)/(x^2+1)) + (\arctan(e^x))^{1/3}$ is an elementary function.

7 The anti derivative of the sinc function is called the **sine-integral**

$$Si(x) = \int_0^x \frac{\sin(t)}{t}\,dt\ .$$

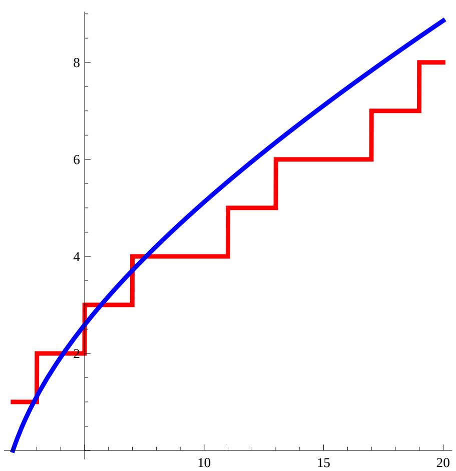
The function $Si(x)$ is not an elementary function.



8 The **offset logarithmic integral** is defined as

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log(t)}$$

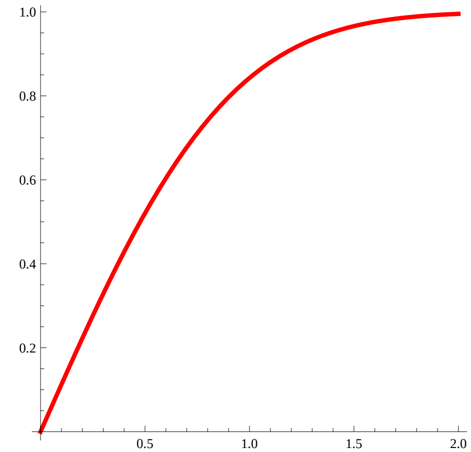
It is a specific anti-derivative. It is a good approximation of the number of prime numbers less than x. The graph below illustrates this. The second stair graph shows the number $\pi(x)$ of primes below x. For example, $\pi(10) = 4$ because $2, 3, 5, 7$ are the only primes below it. The function Li is not an elementary function.



9 The **error function**

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

is important in statistics. It is not an elementary function.



The Mathematica command "Integrate" uses about 500 pages of Mathematica code and 600 pages of C code. [1] Before software was doing this, tables of integrals like Gradshteyn and Ryzhik's work were used. This 1200 page book is still useful and contains some integrals, which computer algebra systems have trouble with.

Numerical evaluation

What do we do when we have can not find the integral analytically? We can still compute it numerically. Here is an example: the function $\sin(\sin(x))$ also does not have an elementary anti-derivative. But you could compute the integral numerically with a computer algebra system like Mathematica:

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NIntegrate[ Sin[Sin[x]],{x,0,10}]
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Pillow problems

We do not assign homework over spring break but if you have time, here are some integration riddles. We will learn techniques to deal with them. If you can not crack them, no problem. Maybe pick one or two and keep thinking about it over spring break. They make also good pillow problems, problems to think about while falling asleep. Try it. Sometimes, you might know the answer in the morning. Maybe you can guess a function which has $f(x)$ as a derivative.

1 $f(x) = \log(x)/x$.

2 $f(x) = \frac{1}{x^4-1}$.

3 $f(x) = \tan^2(x)$.

4 $f(x) = \cos^4(x)$.

5 $f(x) = \frac{1}{x\log(x)}$.

[1] http://reference.wolfram.com/legacy/v3/MainBook/A.9.5.html