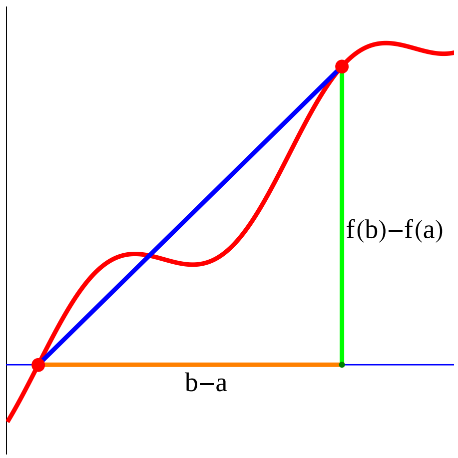
Lecture 16: Mean value theorem

In this lecture, we look at the **mean value theorem** and a special case called **Rolle's theorem**. It is important later when we study the fundamental theorem of calculus. Unlike the intermediate value theorem which applied for continuous functions, the mean value theorem involves derivatives:

> **Mean value theorem**: For a differentiable function f and an interval (a, b), there exists a point p inside the interval, such that
> $$f'(p) = \frac{f(b) - f(a)}{b - a} .$$



Here are a few examples which illustrate the theorem:

1 Verify with the mean value theorem that the function $f(x) = x^2 + 4\sin(\pi x) + 5$ has a point where the derivative is 1.
Solution. Since $f(0) = 5$ and $f(1) = 6$ we see that $(f(1) - f(0))/(1 - 0) = 5$.

2 Verify that the function $f(x) = 4\arctan(x)/\pi - \cos(\pi x)$ has a point where the derivative is 3.
Solution. We have $f(0) = -1$ and $f(1) = 2$. Apply the mean value theorem.

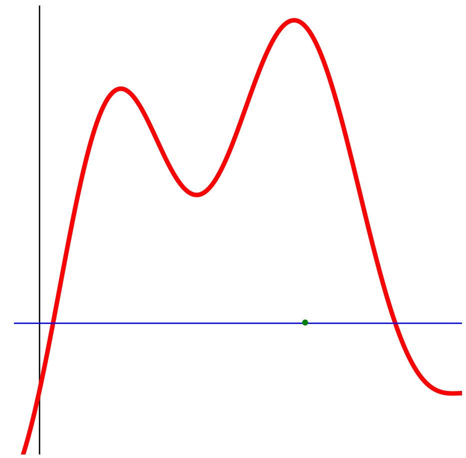
3 A biker drives with velocity $f'(t)$ at position $f(b)$ at time b and at position a at time a. The value $f(b) - f(a)$ is the distance traveled. The fraction $[f(b) - f(a)]/(b - a)$ is the average speed. The theorem tells that there was a time when the bike had exactly the average speed.

4 The function $f(x) = \sqrt{1 - x^2}$ has a graph on $(-1, 1)$ on which every possible slope is taken.
Solution: We can see this with the intermediate value theorem because $f'(x) = x/\sqrt{1 - x^2}$ gets arbitrary large near $x = -1$ or $x = 1$. The mean value theorem shows this too because we can take intervals $[a, b] = [-1, -1 + c]$ for which $[f(b) - f(a)]/(b - a) = f(-1 + c)/ci \sim \sqrt{c}/c = 1/\sqrt{c}$ gets arbitrary large.

Why is the theorem true? The function $h(x) = f(a) + cx$, where $c = (f(b) - f(a)/(b - a)$ also connects the beginning and end point. The function $g(x) = f(x) - h(x)$ has now the property that $g(a) = g(b)$. If we can show that for such a function, there exists x with $g'(x) = 0$, then we are

done. By tilting the picture, we have reduced the statement to a special case which is important by itself:

> **Rolle's theorem**: If $f(a) = f(b)$ and f is differentiable, then there exists a critical point p of f in the interval (a, b).



Here is the proof: If it were not true, then $f'(x) \neq 0$ and we would have $f'(x) > 0$ everywhere or $f'(x) < 0$ everywhere. The monotonicity would mean however that $f(b) > f(a)$ or $f(b) < f(a)$.

Here is a second proof: Fermat's theorem assures that there is a local maximum or local minimum of f in (a, b). At this point the derivative is zero. This means $f'(x) = 0$.

We have also seen a related fact that if f is continuous and $f(a) = f(b)$ then there is a local maximum or local minimum in the interval (a, b). This fact is more general and applies to every continuous function. The derivative does not need to exist.

5 There is a point in $[0, 1]$ where $f'(x) = 0$ with $f(x) = x(1 - x^2)(1 - \sin(\pi x))$. **Solution:** We have $f(0) = f(1) = 0$. Use Rolle's theorem.

6 Show that the function $f(x) = \sin(x) + x(\pi - x)$ has a critical point $[0, \pi]$. **Solution:** The function is nonnegative and zero at $0, \pi$.It is also differentiable and so by Rolle's theorem there is a critical point. Remark. We can not use Rolle's theorem to show that there is a local maximum even so the extremal value theorem assures us that this exist.

7 Verify that the function $f(x) = 2x^3 + 3x^2 + 6x + 1$ has only one real root. **Solution:** There is at least one real root by the intermediate value theorem: $f(-1) = -4, f(1) = 12$. Assume there would be two roots. Then by Rolle's theorem there would be a value x where $g(x) = f'(x) = 6x^2 + 6x + 6 = 0$. But there is no root of g. [The graph of g minimum at $g'(x) = 6 + 12x = 0$ which is 1/2 where $g(1/2) = 21/2 > 0$.]

Who was the first to find the **mean value theorem**? It is not so clear. Joseph Louis Lagrange is one candidate. Also Augustin Louis Cauchy (1789-1857) is credited for a modern formulation of the theorem.

Joseph Louis Lagrange, 1736-1813.

Augustin Louis Cauchy, 1789-1857.

What about **Michel Rolle**? He lived from 1652 to 1719, mostly in Paris. No picture of him seems available. Rolle also introduced the n'th root notation like when writing the cube root as $\sqrt[3]{x}$.

Homework

1 The function $f(x) = 1 - |x|$ satisfies $f(-1) = f(1) = 0$ but there is no point where $f'(x) = 0$. Is this a counter example to Rolle's theorem?

2 Use Rolle's theorem and the intermediate value theorem to show that the function $f(x) = x^3 + 3x + 1$ has exactly one root. You do not have to find the root.

3 We look at the function $f(x) = \log|x| + \sin(x)$ on the positive real line
a) Use the **mean value theorem** to assure the there is a p where $f'(p) = 1000$.

b) Use the **intermediate value theorem** applied to the function $f'(x)$ to assure the

same.

4 **Cauchy's mean value theorem** states that for any two differentiable function and any interval (a, b), there exists c for which $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$. We want to prove this here. Define the function $h(x) = (f(b)-f(a))(g(x)-g(a))-(g(b)-g(a))(f(x)-f(a))$.
a) Verify that $h(a) = h(b) = 0$.

b) Compute $h'(x)$.

c) Use Rolle's theorem to verify that there is a c for which $h'(c) = 0$.

5 Given the function $f(x) = x\sin(x)$ and the function $g(x) = \cos(x)$. Verify (using Cauchy's mean value theorem from the previous problem) that there is a point $p \in (0, \pi/2)$ for which $f'(p)/g'(p) = -\pi/2$. You do not have to find the point.