Lecture 15: Review for first midterm

Major points

A function is **continuous**, if whenever x, y are close, also $f(x), f(y)$ are close. Formally, for every a there exists $b = f(a)$ such that $\lim_{x \to a} f(x) = b$ for every a. The Intermediate value theorem: $f(a) > 0, f(b) < 0$ implies f having a root in (a, b).

If $f'(x) = 0$ and $f''(x) > 0$ then x is a local minimum. If $f'(x) = 0$ and $f''(x) < 0$ then x is a local maximum. To find **global extrema**, compare local extrema and boundary values.

If $f' > 0$ then f is increasing, if $f' < 0$ it is decreasing. If $f''(x) > 0$ it is **concave up**, if $f''(x) < 0$ it is **concave down**. If $f'(x) = 0$ then f has a horizontal tangent.

Hôptial's theorem tells that limits $\lim_{x \to p} f(x)/g(x)$, where $f(p) = g(p) = 0$ or $f(p) = g(p) = \infty$ with $g'(p) \neq 0$ are given by $f'(p)/g'(p)$.

With $Df(x) = (f(x+h) - f(x))/h$ and $S(x) = h(f(h) + f(2h) + ...f(kh))$ we have $SDf(kh) = f(kh) - f(0)$ and $DS(f(kh)) = f(kh)$. This is a preliminary fundamental theorem of calculus.

Roots of $f(x)$ with $f(a) < 0, f(b) > 0$ can be obtained by the dissection method by applying the **Newton map** $T(x) = x - f(x)/f'(x)$ again and again.

Algebra reminders

Healing:	$(a+b)(a-b) = a^2 - b^2$ or $1 + a + a^2 + a^3 + a^4 = (a^5 - 1)/(a - 1)$
Denominator:	$1/a + 1/b = (a+b)/(ab)$
Exponential:	$(e^a)^b = e^{ab}$, $e^a e^b = e^{a+b}$, $a^b = e^{b \log(a)}$
Logarithm:	$\log(ab) = \log(a) + \log(b)$. $\log(a^b) = b \log(a)$
Trig functions:	$\cos^2(x) + \sin^2(x) = 1, \sin(2x) = 2\sin(x)\cos(x), \cos(2x) = \cos^2(x) - \sin^2(x)$
Square roots:	$a^{1/2} = \sqrt{a}$, $a^{-1/2} = 1/\sqrt{a}$

Important functions

Polynomials	$x^3 + 2x^2 + 3x + 1$	Exponential	$5e^{3x}$
Rational functions	$(x+1)/(x^3 + 2x + 1)$	Logarithm	$\log(3x)$
Trig functions	$2\cos(3x)$	Inverse trig functions	$\arctan(x)$

Important derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$f(x) = x^n$	nx^{n-1}	$f(x) = \sin(ax)$	$a\cos(ax)$
$f(x) = e^{ax}$	ae^{ax}	$f(x) = \tan(x)$	$1/\cos^2(x)$
$f(x) = \cos(ax)$	$-a\sin(ax)$	$f(x) = \log(x)$	$1/x$

Differentation rules

Addition rule	$(f+g)' = f' + g'$.	Quotient rule	$(f/g)' = (f'g - fg')/g^2$.
Scaling rule	$(cf)' = cf'$.	Chain rule	$(f(g(x))' = f'(g(x))g'(x)$.
Product rule	$(fg)' = f'g + fg'$.	Easy rule	simplify before deriving

Extremal problems

1 Build a fence of length $x + 2y = 12$ with dimensions x and y with maximal area $A = xy$.

2 Find the largest area $A = 4xy$ of a rectangle with vertices $(x, y), (-x, y), (-x, -y), (x, -y)$ inscribed in the ellipse $x^2 + 2y^2 = 1$.

3 Which isosceles triangle of height h and base $2x$ and area $xh = 1$ has minimal circumference $2x + 2\sqrt{x^2 + h^2}$?

4 Where is the distance $\sqrt{x^2 + y^2}$ of the parabola $y = x^2 - 2$ to the point $(0, 0)$ minimal?

5 A cone of height $h = 1 + x$ and radius $r = \sqrt{1 - x^2}$ is tightly enclosed by a unit sphere centered at height x. Maximize the volume $r^2 \pi h/3$ of the cone.

6 Maximize $f(x) = \sin(x)$ on $[0, \pi]$.

Limit examples

$\lim_{x \to 0} \sin(x)/x$	l'Hopital 0/0	$\lim_{x \to 1} (x^2 - 1)/(x + 1)$	heal directly
$\lim_{x \to 0} (1 - \cos(x))/x^2$	l'Hopital 0/0 twice	$\lim_{x \to \infty} \exp(x)/(1 + \exp(x))$	l'Hopital
$\lim_{x \to 0} x \log(x)$	l'Hopital ∞/∞	$\lim_{x \to 0} (x + 1)/(x + 5)$	no work necessary

Important things

Summation and taking differences is at the hart of calculus
The 3 major types of discontinuities are jump, oscillation, infinity
Dissection and Newton methods are algorithms to find roots. Disection needs continuity, Newton needs
The fundamental theorem of trigonometry is $\lim_{x \to 0} \sin(x)/x = 1$.
The derivative is the limit $Df(x) = [f(x+h) - f(x)]/h$ as $h \to 0$. It is called rate of change.
The rule $D(1+h)^{x/h} = (1+h)^{x/h}$ leads to $\exp'(x) = \exp(x)$.

More Examples

1 Is $1/\log|x|$ continuous at $x = 0$. Answer: yes

2 Is $\log(1/|x|)$ continuous at $x = 0$. Answer: no

3 Find $\lim_{x \to 1} (x^{1/3} - 1)/(x^{1/4} - 1)$. Answer: 4/3.

4 Find $\lim_{x \to 1} \sin(5x - 5)/(x - 1)$. Answer: 5.

5 Find $\lim_{x \to 2} \frac{3 - \sqrt{7 + x}}{x - 2}$. Answer $-1/6$

6 Find $\arcsin'(5x^2)$. Answer: $10x(1 - 25x^4)^{-1/2}$