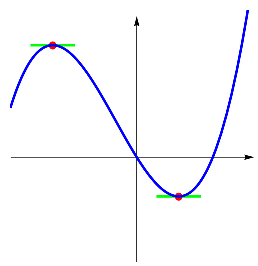
# Lecture 11: Local extrema

Today we look at the problem to find extrema. We want to maximize nice quantities and minimize unpleasant ones. Extremizing quantities is also the most important principle nature follows. Important laws in physics like Newtons law, equations describing light, or matter can be based on the principle of extremization. The most important intuitive insight is that at maxima or minima the tangent to the graph needs to be horizontal. This leads to a zero derivative and the notion of critical points:

A point $x_0$ is a **critical point** of a differentiable function $f$ if $f'(x_0) = 0$.

In some textbooks, critical points include points where $f'$ is not defined. [1] We do here **not** include these points in the list of critical points. They are points outside the domain of definition of $f'$ and will be treated separately.

1 Find the critical points of the function $f(x) = x^3 + 3x^2 - 24x$. **Solution:** we compute the derivative as $f'(x) = 3x^2 + 6x - 24$. The roots of $f'$ are $2, -4$.



A point is called a **local maximum** of $f$, if there exists an interval $U = (p-a, p+a)$ around $p$, such that $f(p) \geq f(x)$ for all $x \in U$. A **local minimum** is a local maximum of $-f$. Local maxima and minima together are called **local extrema**.

2 The point $x = 0$ is a local maximum for $f(x) = \cos(x)$. The reason is that $f(0) = 1$ and $f(x) < 1$ nearby.

3 The point $x = 1$ is a local minimum for $f(x) = (x-1)^2$. The function is zero at $x = 1$ and positive everywhere else.

**Fermat:** If $f$ is differentiable and has a local extremum at $x$, then $f'(x) = 0$.

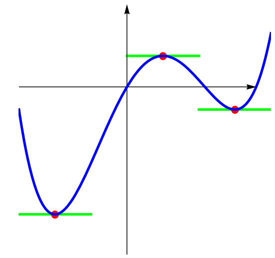
Why? Assume the derivative $f'(x) = c$ is not zero. We can assume $c > 0$ otherwise replace $f$ with $-f$. By the definition of limits, for some large enough $h$, we have $f(x+h) - f(x)/h \geq c/2$. But this means $f(x+h) \geq f(x) + hc/2$ and $x$ can not be a local maximum. Since also $(f(x) - f(x-h))/h \geq c/2$ for small enough $h$, we also have $f(x-h) \leq f(x) - hc/2$ and $x$ can not be a local minimum.

[1] Important definitions have to be simple

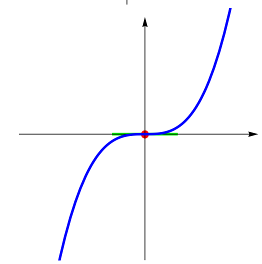




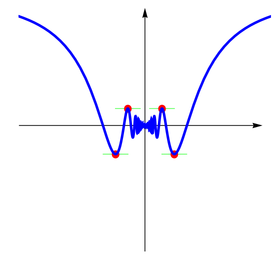
4 The derivative of $f(x) = 72x - 30x^2 - 8x^3 + 3x^4$ is $f'(x) = 72 - 60x - 24x^2 + 12x^3$ By plugging in integers (calculus teachers like integer roots because students like integer roots!) we can guess the roots $x = 1, x = 3, x = -2$ and see $f'(x) = 12(x-1)(x+2)(x-3)$. The critical points are $1, 3, -2$.



5 We have already seen that $f'(x) = 0$ does not assure that $x$ is a local extremum. The function $f(x) = x^3$ is a counter example. It satisfies $f'(0) = 0$ but 0 is not a local extremum. It is an example of an **inflection point**, a point where $f''$ changes sign.



6 Lets look at one nasty example. The function $f(x) = x\sin(1/x)$ is continuous at 0 but there are infinitely many critical points near 0.



If $f''(x) > 0$, then the graph of the function is concave up. If $f''(x) < 0$ then the graph of the function is concave down.

**Second derivative test**. If $x$ is a critical point of $f$ and $f''(x) > 0$, then $f$ is a local minimum. If $f''(x) < 0$, then $f$ is a local maximum.

If $f''(x_0) > 0$ then $f'(x)$ is negative for $x < x_0$ and positive for $f'(x) > x_0$. This means that the function decreases left from the critical point and increases right from the critical point. Similarly, if $f''(x_0) < 0$ then $f'(x)$ is positive for $x < x_0$ and $f'(x)$ is positive for $x > x_0$. This means that the function increases left from the critical point and increases right from the critical point.

7 The function $f(x) = x^2$ has one critical point at $x = 0$. Its second derivative is 2 there.

8 Find the local maxima and minima of the function $f(x) = x^3 - 3x$ using the second derivative test. **Solution:** $f'(x) = 3x^2 - 3$ has the roots $1, -1$. The second derivative $f''(x) = 6x$ is negative at $x = -1$ and positive at $x = 1$. The point $x = -1$ is therefore a local maximum and the point $x = 1$ is a local minimum.

9 Find the local maxima and minima of the function $f(x) = \cos(\pi x)$ using the second derivative test.

10 For the function $f(x) = x^5 - x^3$, the second derivative test is inconclusive at $x = 0$. Can you nevertheless see the critical points?

11 Also for the function $f(x) = x^4$, the second derivative test is inconclusive at $x = 0$. The second derivative is zero. Can you nevertheless see whether the critical point 0 is local maximum of local minimum?

Finally, lets look at an example, where we can practice more the chain rule.

12 Find the critical points of $f(x) = 4\arctan(x) + x^2$. **Solution**. The derivative is

$$f'(x) = \frac{4}{1+x^2} + 2x = \frac{2x + 2x^3 + 4}{1+x^2} \, .$$

We see that $x = -1$ is a critical point. There are no other roots of $2x + 2x^3 + 4 = 0$. How did we get the derivative of arctan again? Differentiate

$$\tan(\arctan(x)) = x$$

and write $u = \arctan(x)$ :

$$\frac{1}{\cos^2(u)} \arctan'(x) = 1 \, .$$

Use the identity $1 + \tan^2(u) = 1/\cos^2(u)$ to write this as

$$(1 + \tan^2(u)) \arctan'(x) = 1 \, .$$

But $\tan(u) = \tan(\arctan(x)) = x$ so that $\tan^2(u) = x^2$. And we have

$$(1 + x^2) \arctan'(x) = 1 \, .$$

Now solve for $\arctan'(x)$:

$$\arctan'(x) = \frac{1}{1+x^2} \, .$$



# Homework

1 Find all critical points for the following functions. If there are infinitely many, indicate their structure. For $f(x) = \cos(x)$ for example, the critical points can be written as $\pi/2 + k\pi$, where $k$ is an integer.
a) $f(x) = x^4 - 3x^2$.
b) $f(x) = 3 + \sin(\pi x)$
c) $f(x) = \exp(-x^2)x^2$.
d) $f(x) = \cos(\sin(x))$

2 Find all the maxima and minima using the second derivative test:
a) $f(x) = x\log(x)$, where $x > 0$.
b) $f(x) = 1/(1 + x^2)$
c) $f(x) = x^2 - 2x + 1$.
d) $f(x) = 2x\tan(x)$, where $-\pi/2 < x < \pi/2$

3 Verify that a cubic equation $f(x) = x^3 + ax^2 + bx + c$ always has an inflection point, a point where $f''(x)$ changes sign.
**Hint.** Remember the wobbling table!

4 Depending on $c$, the function $f(x) = x^4 - cx^2$ has either one or three critical points. Find these points for a general $c$ and use the second derivative test to see whether they are maxima or minima. The answer will depend on $c$. Where does the answer change?

5 a) Engineer a function which has exactly 2 local maximum and 1 local minimum.
b) Find a function which has exactly 2 local maxima and no local minimum.