Lecture 10: The chain rule

How do we take the derivative of a composition of functions? It is the chain rule which will allow us to compute derivatives like for $f(x) = \sin(x^7)$ which is a composition of two functions $f(x) = x^7$ and $g(x) = \sin(x)$. The product rule does not work here. The functions are "chained", we evaluate first x^7 then apply sin to it. In order to differentiate, we the derivative of the first function we evaluate x^7 then multiply this with the derivative of the function sin at x^7. The answer is $7x^6 \cos(x^7)$.

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) .$$

The chain rule follows from the identity

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + (g(x+h) - g(x))) - f(g(x))]}{[g(x+h) - g(x)]} \cdot \frac{[g(x+h) - g(x)]}{h} .$$

Write $H(x) =$ g(x+h)-g(x) in the first part on the right hand side

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + H) - f(g(x))]}{H} \cdot \frac{g(x+h) - g(x)}{h} .$$

As $h \to 0$, we also have $H \to 0$ and the first part goes to $f'(g(x))$ and the second factor has $g'(x)$ as a limit. The chain rule is one reason why classical calculus is so elegant: $D(f(g)) = (D_H f)(g(x))D(g(x))$. The h has changed.

1 Find the derivative of $f(x) = (4x-1)^{17}$. **Solution** The inner function is $g(x) = 4x - 1$. It has the derivative 4. We get therefore $f'(x) = 17(4x-1)^6 \cdot 4 = 28(4x-1)^6$. Remark. We could have expanded out the power $(4x-1)^{17}$ first and avoided the chain rule. Avoiding the **chain rule** is called the **pain rule**.

2 Find the derivative of $f(x) = \sin(\pi \cos(x))$ at $x = 0$. **Solution**: applying the chain rule gives $\cos(\pi \cos(x)) \cdot (-\pi \sin(x))$.

3 For linear functions $f(x) = ax + b, g(x) = cx + d$, the chain rule can readily be checked. We have $f(g(x)) = a(cx+d) + b = acx + ad + b$ which has the derivative ac. Indeed this is the definition of f times the derivative of g. You can convince you that the chain rule is true also from this example since if you look closely at a point, then the function is close to linear.

One of the cool applications of the chain rule is that we can compute derivatives of inverse functions:

4 Find the derivative of the natural logarithm function $\log(x)$ [1] **Solution** Differentiate the identity $\exp(\log(x)) = x$. On the right hand side we have 1. On the left hand side the chain rule gives $\exp(\log(x))\log'(x) = x\log'(x) = 1$. Therefore $\log'(x) = 1/x$.

[1]We always write $\log(x)$ for the natural log. The ln notation is old fashioned and only used in obscure places like calculus books and calculators from the last millenium.

$\frac{d}{dx}\log(x) = 1/x$.

Denote by $\arccos(x)$ the inverse of $\cos(x)$ on $[0, \pi]$ and with $\arcsin(x)$ the inverse of $\sin(x)$ on $[-\pi/2, \pi/2]$.

5 Find the derivative of $\arcsin(x)$. **Solution**. We write $x = \sin(\arcsin(x))$ and differentiate.

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}} .$$

6 Find the derivative of $\arccos(x)$. **Solution**. We write $x = \cos(\arccos(x))$ and differentiate.

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}} .$$

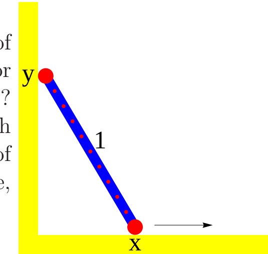
7 $f(x) = \sin(x^2 + 3)$. Then $f'(x) = \cos(x^2 + 3)2x$.

8 $f(x) = \sin(\sin(\sin(x)))$. Then $f'(x) = \cos(\sin(\sin(x)))\cos(\sin(x))\cos(x)$.

Why is the chain rule called "chain rule". The reason is that we can chain even more functions together.

9 Lets compute the derivative of $\sin(\sqrt{x^5 - 1})$ for example. **Solution:** This is a composition of three functions $f(g(h(x)))$, where $h(x) = x^5 - 1$, $g(x) = \sqrt{x}$ and $f(x) = \sin(x)$. The chain rule applied to the function $\sin(x)$ and $\sqrt{x^5-1}$ gives $\cos(\sqrt{x^5-1})\frac{d}{dx}\sqrt{x^5-1}$. Apply now the chain rule again for the derivative on the right hand side.

10 Here is the famous **falling ladder problem**. A stick of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point $(0, y)$ on the wall with $(x, 0)$ on the floor. We want to express y as a function of x. We have $y = f(x) = \sqrt{1-x^2}$. Taking the derivative, assuming $x' = 1$ gives $f'(x) = -2x/\sqrt{1-x^2}$.



In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle** $\theta = \arcsin((2v^2/(3g))^{2/3})$, where g is the gravitational acceleration and $v = x'$ is the velocity.

11 For the brave: find the derivative of $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(x)))))))$.

Homework

1 Find the derivatives of the following functions
a) $f(x) = \sin(\sqrt{x}$
b) $f(x) = \tan(1/x^7)$
c) $f(x) = \exp(1/(1+x))$
d) $(2 + \sin(x))^{-5}$

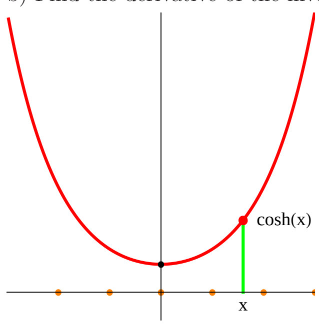
2 Find the derivatives of the following functions at $x = 1$.
a) $f(x) = x^8 \log(x)$. (log is natural log)
b) $\sqrt{x^5 + 1}$

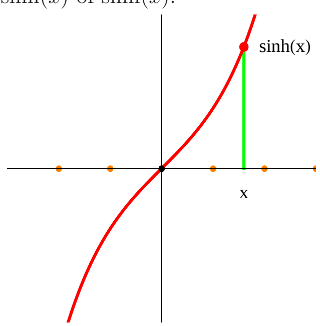
3 a) Find the derivative of $f(x) = 1/x$ by differentiating the identity $xf(x) = 1$.
b) Find the derivative of $f(x) = \text{arccot}(x)$ by differentiating $\cot(\text{arccot}(x)) = x$.

4 a) Find the derivative of $f(x) = \sqrt{x}$ by differentiating the identity $f(x)^2 = x$.
b) Find the derivative of $f(x) = x^{m/n}$ by differentiating the identity $f(x)^n = x^m$.

The function $f(x) = [\exp(x) + \exp(-x)]/2$ is called $\cosh(x)$.
The function $f(x) = [\exp(x) - \exp(-x)]/2$ is called $\sinh(x)$.
They are called **hyperbolic cosine** and **hyperbolic sine**. The first is even, the second is odd. You can see directly using $\exp'(x) = \exp(x)$ and $\exp'(-x) = -\exp(-x)$ that $\sinh'(x) = \cosh(x)$ and $\cosh'(x) = \sinh(x)$. Furthermore $\exp = \cosh + \sinh$ writes exp as a sum of an even and odd function.

5 a) Find the derivative of the inverse $\text{arccosh}(x)$ of $\cosh(x)$.
b) Find the derivative of the inverse $\text{arcsinh}(x)$ of $\sinh(x)$.





The cosh function is the shape of a chain hanging at two points. The shape is the hyperbolic cosine.