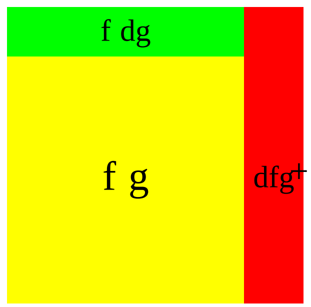
Lecture 9: The product rule

In this lecture, we look at the derivative of a product of functions. The product rule is also called **Leibniz rule** named after **Gottfried Leibniz**, who found it in 1684. It is a very important rule because it allows us to differentiate many more functions. If we wanted to compute the derivative of $f(x) = x\sin(x)$ for example, we would have to get under the hood of the function and compute the limit $\lim(f(x+h) - f(x))/h$. We are too lazy for that. Lets start with the identity

$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h) - f(x)] \cdot g(x+h) + f(x) \cdot [g(x+h) - g(x)]$$



which can be written as $D(fg) = Dfg^+ + fDg$ with $g^+(x) = g(x+h)$. This **quantum Leibniz rule** can also be seen geometrically: the rectangle of area $(f+df)(g+dg)$ is the union of rectangles with area $f \cdot g$, $f \cdot dg$ and $df \cdot g^+$. Divide this relation by h to see

$$\frac{[f(x+h) - f(x)]}{h} \cdot g(x+h) \;\to\; f'(x) \cdot g(x)$$
$$f(x) \cdot \frac{[g(x+h) - g(x)]}{h} \;\to\; f(x) \cdot g'(x) \; .$$

We get the extraordinarily important **product rule**:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \; .$$

Remark: the Quantum Leibniz rule can also be seen in the **Babylonian calculus** developed in the first hour. Take $h = 1$ and compute $Dx^2 = D(xx) = Dx(x+1) + xDx = 1 \cdot (x+1) + x \cdot 1 = 2x+1$. Indeed, we have seen that summing up all odd numbers $2x+1$ gives the squares x^2.

1 Find the derivative function $f'(x)$ for $f(x) = x^3\sin(x)$. **Solution:** We know how to differentiate x^3 and $\sin(x)$ so that $f'(x) = 3x^2\sin(x) + x^3\cos(x)$.

2 While we know

$$\frac{d}{dx}x^5 = 5x^4$$

lets compute this with the Leibniz rule and write $x^5 = x^3 \cdot x^2$. We have

$$\frac{d}{dx}x^3 = 3x^2, \frac{d}{dx}x^2 = 2x \; .$$

The Leibniz rule gives us $d/dx^5 = 3x^4 + 2x^4 = 5x^4$.

3 Lets look at a few derivatives related to functions where we know the answer already but where we can check things using the product formula:

- $\frac{d}{dx}(x^3 \cdot x^5)$
- $\frac{d}{dx}e^{3x}e^{5x}$
- $\frac{d}{dx}\sqrt{x}/\sqrt{x}$
- $\frac{d}{dx}\sin(x)\cos(x)$

Before we look at the quotient rule which allows to differentiate $f(x)/g(x)$ we can also write the later as $f(x) \cdot 1/g(x)$ and use a rule telling us how to differentiate $1/g(x)$. This is the **reciprocal rule**:

If $g(x) \neq 0$, then
$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2} \; .$$

In order to see this $h = 1/g$ and differentiate the equation $1 = g(x)h(x)$ on both sides. The product rule gives $0 = g'(x)h(x) + g(x)h'(x)$ so that $h'(x) = -h(x)g'(x)/g(x) = -g'(x)/g^2(x)$.

4 Find the derivative of $f(x) = 1/x^4$. **Solution**: $f'(x) = -4x^3/x^8 = -4/x^5$. The same computation shows that $\frac{d}{dx}x^n = nx^{n-1}$ holds for all integers n.

The formula $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$ holds for all integers n.

The **quotient rule** is obtained by applying the product rule to $f(x) \cdot (1/g(x))$ and using the reciprocal rule:

If $g(x) \neq 0$, then
$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{[f'(x)g(x) - f(x)g'(x)]}{g^2(x)} \; .$$

5 Find the derivative of $f(x) = \tan(x)$. **Solution**: because $\tan(x) = \sin(x)/\cos(x)$ we have

$$\tan'(x) = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} \; .$$

6 Find the derivative of $f(x) = \frac{2-x}{x^2+x^4+1}$. **Solution**. We apply the quotient rule and get $[(-1)x^2 + x^4 + 1 + (2-x)(2x+4x^3)]/(x^2+x^4+1)$.

Here are some more problems with solutions:

7 Find the second derivative of $\tan(x)$. **Solution**. We have already computed $\tan'(x) = 1/\cos^2(x)$. Differentiate this again with the quotient rule gives

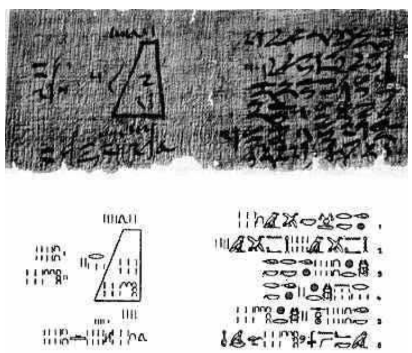
$$\frac{-\frac{d}{dx}\cos^2(x)}{\cos^4(x)} .$$

We still have to find the derivative of $\cos^2(x)$. The product rule gives $\cos(x)\sin(x) + \sin(x)\cos(x) = 2\cos(x)\sin(x)$. Our final result is

$$2\sin(x)/\cos^3(x) .$$

8 A cylinder has volume $V = \pi r^2 h$, where r is the radius and h is the height. Assume the radius grows like $r(t) = 1 + t$ and the height shrinks like $1 - \sin(t)$. Does the volume grow or decrease at $t = 0$?
Solution: The volume $V(t) = \pi(1+t)^2(1-\sin(t))$ is a product of two functions $f(t) = \pi(1+t)^2$ and $g(t) = (1-\sin(t)$. We have $f(0) = 1, g'(0) = 2, f'(0) = 2, g(0) = 1$. The product rule gives gives $V'(0) = \pi 1 \cdot (-1) + \pi 2 \cdot 1 = \pi$. The volume increases in volume at first.

9 On the **Moscow papyrus** dating back to 1850 BC, the general formula $V = h(a^2 + ab + b^2)/3$ for a truncated pyramid with base length a, roof length b and height h appeared. Assume $h(t) = 1 + \sin(t), a(t) = 1 + t, b(t) = 1 - 2t$. Does the volume of the truncated pyramid grow or decrease at first? **Solution**. We could fill in $a(t), b(t), h(t)$ into the formula for V and compute the derivative using the product rule. A bit faster is to write $f(t) = a^2 + ab + b^2 = (1+t)^2 + (1-3t)^2 + (1+t)(1-3t)$ and note $f(0) = 3, f'(0) = -6$ then get from $h(t) = (1+\sin(t))$ the data $h(0) = 1, h'(0) = 1$. So that $V'(0) = (h'(0)f(0) - h(0)f'(0))/3 = (1 \cdot 3 - 1(-6))/3 = -1$. The pyramid shrinks in volume at first.



10 We pump up a balloon and let it fly. Assume that the thrust increases like t and the resistance decreases like $1/\sqrt{1-t}$ since the balloon gets smaller. The distance traveled is $f(t) = t/\sqrt{1-t}$. Find the velocity $f'(t)$ at time $t = 0$.

Homework

1 Find the derivatives of the following functions:
a) $f(x) = \sin(3x)\cos(10x)$.
b) $f(x) = \sin^2(x)/x^2$.
c) $f(x) = x^4 \sin(x)\cos(x)$.
d) $f(x) = 1/\sqrt{x}$.
e) $f(x) = \cot(x) + (1+x)/(1+x^2)$.

2 a) Verify that for $f(x) = g(x)h(x)k(x)l(x)$ the formula $f' = g'hkl + gh'kl + ghk'l + ghkl'$ holds.
b) Verify the following formula for derivative of $f(x) = g(x)^4$: $f'(x) = 4g^3(x)g'(x)$. We will derive this later using the chain rule. Don't use that rule yet.

3 If $f(x) = \text{sinc}(x) = \sin(x)/x$, find its derivative $g(x) = f'(x)$ and then the derivative of $g(x)$. Then evaluate it numerically at $x = 0$.

4 Find the derivative of

$$\frac{\sin(x)}{1 + \cos(x) + \frac{x^4}{1+\cos^2(x)}}$$

at $x = 0$.

5 a) We have already computed the derivative of $f(x) = \sqrt{x}$ in the last homework by directly computing the limit. Lets do it using the product rule. Use part a) of this problem to compute the derivative of

$$g(x) = f(x) \cdot f(x)$$

Use the obtained identity $g'(x) = ...$ to get a formula for $f'(x) = \frac{d}{dx}\sqrt{g(x)}$.
b) Use the same method and the above homework problem 2) in this homework set to compute the derivative of the cube root function $f(x) = x^{1/4}$.

Remark: Also this last problem 5) is a preparation for the chain rule, we see next Monday. Avoid using the chain rule already here.