Lecture 4: Continuity

A **function** f is called **continuous** at a point p if a value $f(p)$ can be found such that $f(x) \to f(p)$ for $x \to p$. A function f is called **continuous on** $[a, b]$ if it is continuous for every point x in the interval $[a, b]$.

In the interior (a, b), the limit needs to exist both from the right and from the left. At the boundary a only the right limit needs to exist and at b only the left limit. Intuitively, a function is continuous if you can **draw the graph of the function without lifting the pencil**. Continuity means that small changes in x results in small changes of $f(x)$.

1 Any polynomial as well as $\cos(x), \sin(x), \exp(x)$ are continuous everywhere. Also the sum and product of such functions is continuous. For example

$$\sin(x^3 + x) - \cos(x^5 + x^3)$$

is continuous everywhere.

2 The function $f(x) = 1/x$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous due to a **pole**. The **division by zero** kills continuity. Remember however that this can be salvaged in some cases like $f(x) = \sin(x)/x$ which is continuous everywhere.

3 The function $f(x) = \log|x|$ is continuous for x different from 0. It is not continuous at 0 because $f(x) \to -\infty$ for $|x| \to 0$. Keep the two examples, $1/x$ and $\log|x|$ in mind.

4 The function $\csc(x) = 1/\sin(x)$ is not continuous at $x = 0, x = \pi, x = 2\pi$ and any multiple of π. It has poles there because $\sin(x)$ is zero there and because we would divide by zero at such points.

5 The function $f(x) = \sin(\pi/x)$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous due to **oscillation**. We can approach $x = 0$ in ways that $f(x_n) = 1$ and such that $f(z_n) = -1$. Just chose $x_n = 2/(4k + 1)$ and $z_n = 2/(4k - 1)$.

6 The **signum function** $f(x) = \text{sign}(x) = \begin{bmatrix} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{bmatrix}$ is not continuous at 0. It is a prototype of a function which has a **jump** discontinuity at 0.

We can refine the notion of continuity and say that a function is **continuous from the right**, if there exists a limit from the right $\lim_{x \downarrow a} f(x) = b$. Similarly a function f can be continuous from the left only. Most of the time we mean with "continuous"= "continuous on the real line".

Rules:
a) If f and g are continuous, then $f + g$ is continuous.
b) If f and g are continuous, then $f * g$ is continuous.
c) If f and g are continuous and if $g > 0$ then f/g is continuous.
d) If f and g are continuous, then $f \circ g$ is continuous.

7 $\sqrt{x^2 + 1}$ is continuous everywhere on the real line.

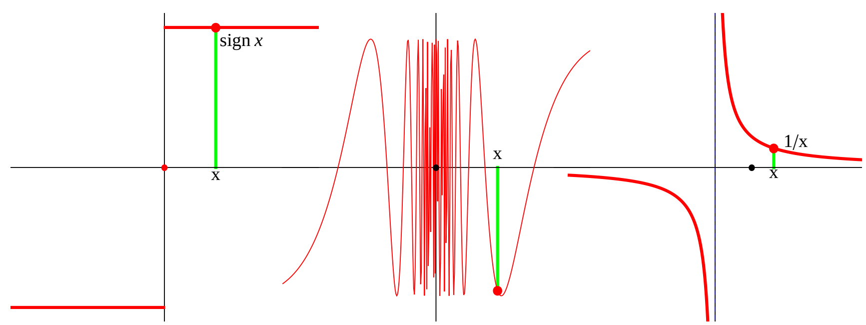
8 $\cos(x) + \sin(x)$ is continuous everywhere.

9 The function $f(x) = \log(|x|)$ is continuous everywhere except at 0. Indeed since for every integer n, we have $f(e^{-n}) = -n$, this can become arbitrarily large for $n \to \infty$ even so e^{-n} converges to 0 for n running to infinity.

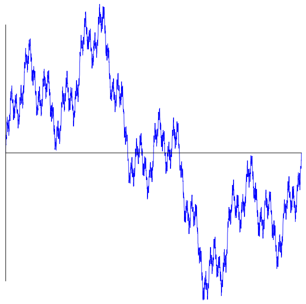
10 While $\log(|x|)$ is not continuous at $x = 0$, the function $1/\log|x|$ is continuous at $x = 0$. Is it continuous everywhere?

11 The function $f(x) = [\sin(x + h) - \sin(x)]/h$ is continuous for every $h > 0$. We will see next week that nothing bad happens when h becomes smaller and smaller and that the continuity will not deteriorate. Indeed, we will see that we get closer and closer to the cos function.

There are three major reasons, why a function is not continuous at a point: it can **jump**, **oscillate** or **escape** to infinity. Here are the prototype examples. We will look at more during the lecture.



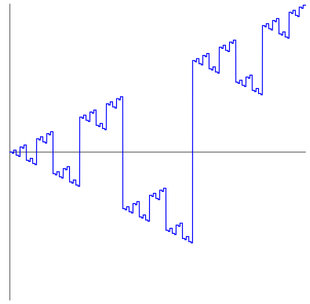
Why do we like continuity? We will see many reasons during this course but for now lets just say that:



A wild continuous function. This Weierstrass function is believed to be a fractal.

"Continuity tames a function.

It can be pretty wild, but not too crazy."



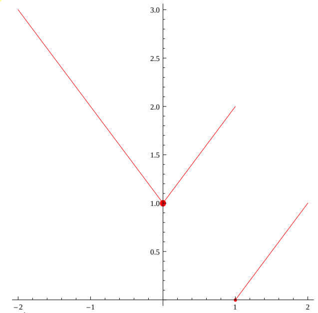
A crazy discontinuous function. It is discontinuous at every point and known to be a fractal.

Continuity will be useful later for extremization. A continuous function on an interval $[a, b]$ has a maximum and minimum. And if a continuous function is negative at some place and positive at an other, there is a point between, where it is zero. These are all useful properties to have and they do not hold if a function is not continuous.

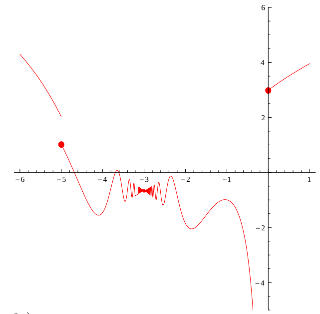
12 **Problem** Determine for each of the following functions, where discontinuities appear:
a) $f(x) = \log(|x^2 - 1|)$
b) $f(x) = \sin(\cos(\pi/x))$
c) $f(x) = \cot(x) + \tan(x) + x^4$
d) $f(x) = x^4 + 5x^2 - 3x + 4$
e) $f(x) = \frac{x^2-4x}{x}$
Solution.
a) $\log(|x|)$ is continuous everywhere except at $x = 0$. Since $x^2 - 1 = 0$ for $x = 1$ or $x = -1$, the function $f(x)$ is continuous everywhere except at $x = 1$ and $x = -1$.
b) The function π/x is continuous everywhere except at $x = 0$. Therefore $\cos(\cos(\pi/x)$ is continuous everywhere except possibly at $x = 0$. We have still to investigate the point $x = 0$ but there, the function $\cos(\pi/x)$ takes values between -1 and 1 for points arbitrarily close to $x = 0$. The function $f(x)$ takes values between $\sin(-1)$ and $\sin(1)$ arbitrarily close to $x = 0$. It is not continuous there.
c) The function x^4 is continuous everywhere. The function $\tan(x)$ is continuous everywhere except at the points $k\pi$. The function $\cot(x)$ is continuous everywhere except at points $\pi/2 + k\pi$. The function f is therefore continuous everywhere except at the point $x = k\pi/2$, multiples of $\pi/2$.
d) The function is a polynomial. We know that polynomials are continuous everywhere.
e) The function is continuous everywhere except at $x = 0$, where we have to look at the function more closely. But we can heal the function by dividing nominator and denominator by x which is possible for x different from 0. The healed function is $f(x) = x - 4$.

Homework

1 On which intervals is the following function continuous?



a)



b)

2 For the following functions, determine the points, where f is not continuous.
a) $f(x) = \cot(1 - x)$
b) $x^3 \cos(1/x)$
c) $\text{sign}(x)/x$
d) $\textbf{sinc}(x) + \sin(x^2) + x^{22} + \log|x|$
e) $\frac{x^2+5x+x^4}{x-1}$
State which kind of discontinuity appears.

3 Construct a function which has a jump discontinuity, an oscillatory one as well as an escape to infinity. Can you construct an example where two of these flaws happen at the same point? Can you even construct an example, where all three happen at the same point?

4 Heal the following functions:
a) $(x^4 - 16)/(x - 2)$
b) $x^5 + x^3/(x^2 + 1)$
c) $((\sin(x))^3 - \sin(x))/\sin(x)$.
d) $(x^3 + 3x^2 + 3x + 1)/(x^2 + 2x + 1)$
e) $(x^{1000} - 1)/(x^{100} - 1)$

5 Are the following function continuous? Break the functions up into simpler functions and analyze each. If you are not sure, also experiment by plotting the functions.
a) $\sin(\frac{1}{2+\sin(x)}) + |\cos(x)| + \frac{\sin(x)}{x} + x^5 + x^3 + 1 + \frac{1}{\exp(x)}$.
b) $\frac{1}{\log|x|} + x^5 - \sin(\sin(\sin(x))) - \exp(\exp(\exp(x)))$