# Lecture 3: Limits

We have seen that functions like $1/x$ are not defined everywhere. Sometimes, however, functions do not make sense at first but can nevertheless be saved. A silly example is $f(x) = x/x$ which is apriori not defined at $x = 0$ because we divide by $x$ but can be "saved" by noticing that $f(x) = 1$ for all $x$ different from 0. Functions often can be continued to "non-allowed" places if we write the function differently. This often involves dividing out a common factor. Lets look at some examples:

1 The function $f(x) = (x^3 - 1)/(x - 1)$ is at first not defined at $x = 1$. However, for $x$ close to 1, nothing really bad happens. We can evaluate the function at points closer and closer to 1 and get closer and closer to 3. We will say $\lim_{x \to 1} f(x) = 3$. Indeed, as you might have noticed already, we have $f(x) = x^2 + x + 1$ by factoring out the term $(x - 1)$. While the function was initially not defined at $x = 1$, we can assign a natural value 3 at the point $x = 1$ so that the graph continues nicely through that point.

Definition. We write $x \to a$ to say that the number $x$ approaches $a$ from either side. A function $f(x)$ has a **limit** at a point $a$ if there exists $b$ such that $f(x) \to b$ for $x \to a$. We write $\lim_{x \to a} f(x) = b$. It should not matter, whether we approach $a$ from the left or from the right. In both cases, we should get the same limiting value $b$.

2 The function $f(x) = \sin(x)/x$ is called $\text{sinc}(x)$. It converges to 1 as $x \to 0$. We can see this geometrically by comparing the side $a = \sin(x)$ of a right angle triangle with a small angle $\alpha = x$ and hypotenuse 1 with the length of the arc between $B, C$ of the unit circle centered at $A$. The arc has length $x$ which is close to $\sin(x)$ for small $x$. Keep this example in mind. It is a crucial one. The fact that the limit of $f(x)$ exists for $x \to 0$ is some important that it is sometimes called the **fundamental theorem of trigonometry**.

3 The quadratic function $f(x) = x^2$ has the property that $f(x)$ approaches 4 if $x$ approaches 2. To evaluate functions at a point, we do not have to take a limit. The function is already defined there. This is important: most points are "healthy". We do not have to worry about limits there. In most cases we see in real applications we only have to worry about limits when the function involves division by 0. For example $f(x) = (x^4 + x^2 + 1)/x$ needs to be investigated carefully at $x = 0$. You see for example that for $x = 1/1000$, the function is slightly larger than 1000, for $x = 1/1000000$ it is larger than one million. There is no rescue here. The limit does not exist at 0.

4 More generally, for all polynomials, the limit $\lim_{x \to a} f(x) = f(a)$ is defined. We do not have to worry about limits, if we deal with polynomials.

5 For all trigonometric polynomials involving sin and cos, the limit $\lim_{x \to a} f(x) = f(a)$ is defined. We do not have to worry about limits if we deal with trigonometric polynomials like $\sin(3x) + \cos(5x)$. The function $\tan(x)$ however has no limit at $x = \pi/2$. No finite value $b$ can be found so that $\tan(\pi/2 + h) \to b$ for $h \to 0$. This is due to the fact that $\cos(x)$ is zero at $\pi/2$. We have $\tan(x)$ goes to $+\infty$ "plus infinity" for $x \searrow \pi/2$ and $\tan(x)$ goes to $-\infty$ for $x \nearrow \pi/2$. In the first case, we approach $\pi/2$ from the right and in the second case from the left.

6 The **cube root** function $f(x) = x^{1/3}$ converges to 0 as $x \to 0$. For $x = 1/1000$ for example, we have $f(x) = 1/10$ for $x = 1/n^3$ the value $f(x)$ is $1/n$. The cube root function is defined everywhere on the real line, like $f(-8) = -2$ and is continuous everywhere.





Why do we worry about limits at all? One of the main reasons will is that we will define the derivative and integral using limits. But we will also use limits to get numbers like $\pi = 3.1415926, .....$ In the next lecture, we will look at the important concept of continuity, which involves limits too.

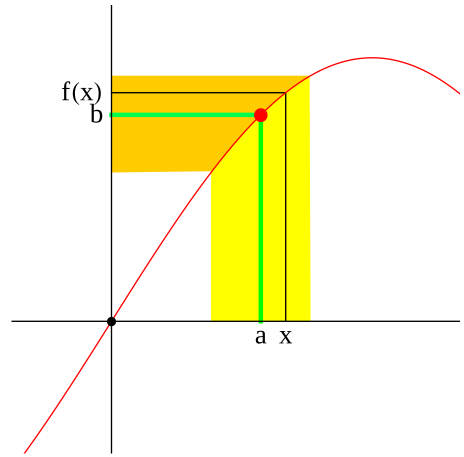


**Figure:** We can test whether a function has the limit $b$ at a point $a$ if for every vertical interval $I$ containing $b$ there exists a horizontal interval $J$ containing $a$ such that if $x$ is in $J$, then $f(x)$ is in $I$. If the function stays bounded, does not oscillate at the point like $\sin(1/x)$ or jump, then the limit exists.

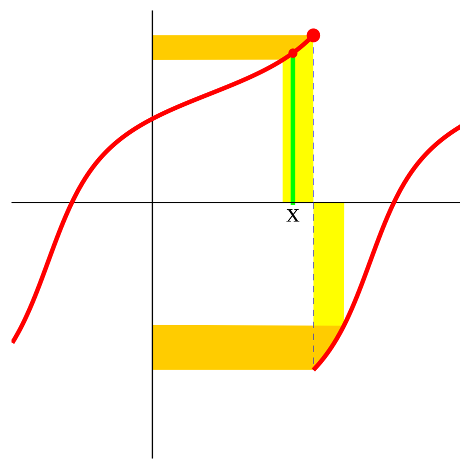


**Figure:** We see here the function $f(x) = \arctan(\tan(x) + 1)$, where *arctan* is the inverse of *tan* giving the angle from the slope. In this case, the limit does not exist for $a = \pi/2$. If we approach this point $a$ from the right, we are always far below the limiting value. The limit exists from the left if we postulate $f(\pi/2) = \pi/2$. Note that $f$ has a priori no value at $x = \pi/2$ because $\tan(x)$ becomes infinite there.

7 **Problem:** Determine from the following functions whether the limits $\lim_{x \to 0} f(x)$ exist. If the limit exists, find it.
a) $f(x) = \cos(x)/\cos(2x)$
b) $f(x) = \tan(x)/x$
c) $f(x) = (x^2 - x)/(x - 1)$
d) $f(x) = (x^4 - 1)/(x^2 - 1)$
e) $f(x) = (x + 1)/(x - 1)$
f) $f(x) = x/\sin(x)$
g) $f(x) = 5x/\sin(6x)$
h) $f(x) = \sin(x)/x^2$
i) $f(x) = \sin(x)/\sin(2x)$
j) $f(x) = \exp(x)/x$

**Solutions:**

a) There is no problem at all at $x = 0$. Both, the nominator and denominator converge to 1. The limit is $\boxed{1}$
b) This is $\text{sinc}(x)/\cos(x)$. There is no problem at $x = 0$ for sinc nor for $1/\cos(x)$. The limit is $\boxed{1}$.
c) We can heal this function. It is the same as $x + 1$ everywhere except at $x = 1$ where it is not defined. But we can continue the simplified function $x + 1$ through $x = 1$. The limit is $\boxed{2}$.
d) We can heal this function. It is the same as $x^2 + 1$. The limit is $\boxed{2}$.
e) There is no problem at $x = 0$. There is mischief at $x = 1$ although but that is far, far away. At $x = 0$, we get $\boxed{1}$.
f) This is the prototype, the fundamental theorem of trig! We know that the limit is $\boxed{1}$.
g) This can be written as $f(x) = (5/6)6x/\sin(6x) = (5/6)\text{sinc}(6x)$. The function $6x/\sin(6x)$ converges to 1 by the fundamental theorem of trigonometry. Therefore the limit is $\boxed{5/6}$.
h) This limit does not exist. It can be written as $\text{sinc}(x)/x$. Because $\text{sinc}(x)$ converges to 1. we are in trouble when dividing again by $x$. $\boxed{\text{There is no limit.}}$
i) We know $\sin(x)/x \to 1$ so that also $\sin(2x)/(2x)$ has the limit 1. If we divide them, see $\sin(x)/\sin(2x) \to 1/2$. The result is $\boxed{1/2}$.
j) The limit does not exist because $\exp(x) \to 1$ but $1/x$ goes to infinity.

# Homework

1 Find the limits of each of the following functions at the point $x \to 0$. You can use the fact that $\sin(x)/x$ has the limit 1 as $x \to 0$.
a) $f(x) = (x^2 - 1)/(x - 1)$
b) $f(x) = \sin(3x)/x$
c) $f(x) = \sin^2(5x)/x^2$
d) $f(x) = \sin(3x)/\sin(5x)$

2 a) Graph of the function

$$f(x) = \frac{(1 - \cos(x))}{x^2} .$$

b) Where is the function $f$ defined? Can you find the limit at the places, where it is not defined?

3 a) Can you see the limit of $g(h) = [f(x+h) - f(x)]/h$ as a function of $h$ at the point $x = 0$ for the function $f(x) = \sin(x)$?
b) Verify that the function $f(x) = \exp_h(x) = (1+h)^{x/h}$ satisfies $[f(x+h) - f(x)]/h = f(x)$.
**Remark.** The exponential function can be defined as $e^x = \exp(x) = \lim_{h \to 0} \exp_h(x)$.

4 Find the limits for $x \to 0$:
a) $f(x) = (x^2 - 2x + 1)/(x - 1)$.
b) $f(x) = 2^x$.
c) $f(x) = 2^{2^x}$.
d) $f(x) = \sin(\sin(x))/\sin(x)$.

5 We explore in this problem the limit of the function $f(x) = x^x$ if $x \to 0$. Can we find a limit? Take a calculator or use Wolfram $\alpha$ and experiment. What do you see when $x \to 0$? Only optional: can you find a explanation for your experiments?