Lecture 2: Functions

A **function** is a rule which assigns to a real number a new real number. The function $f(x) = x^2 - 2x$ for example assigns to the number $x = 4$ the value $4^2 - 8 = 8$. A function is given with a **domain** A, the points where f is defined and a **codomain** B a set of numbers which f can reach. The function $f(x) = x^2 - 2x$ is defined on the entire real axes. While $f(x) \geq -1$, the codomain is the set of real numbers.

Typically, the codomain agrees with the set of real numbers and the domain to be all the numbers, where the function is defined. The function $f(x) = 1/x$ for example is not defined at $x = 0$ so that we chose the domain $A = R \setminus \{0\}$, all numbers except 0. The function $f(x) = 1/x$ takes values in the codomain R. If we choose $A = B$, then $f(x) = 1/x$ reaches every point in B and is invertible. It is its own inverse. Here are a few examples of functions. We will look at them in more detail during the lecture, especially the polynomials, trigonometric functions and exponential function.

identity	$f(x) = x$	power	$f(x) = 2^x$
constant	$f(x) = 1$	exponential	$f(x) = e^x = \exp(x)$
linear	$f(x) = 3x + 1$	logarithm	$f(x) = \log(x) = \exp^{-1}(x)$
quadratic	$f(x) = x^2$	absolute value	$f(x) = \|x\|$
cosine	$f(x) = \cos(x)$	devil comb	$f(x) = \sin(1/x)$
sine	$f(x) = \sin(x)$	bell function	$f(x) = e^{-x^2}$
exponentials	$f(x) = \exp_h(x) = (1 + h)^{x/h}$	witch of Agnesi	$f(x) = \frac{1}{1+x^2}$
logarithms	$f(x) = \log_h(x) = \exp_h^{-1}$	sinc	$\sin(x)/x$

We can build new functions by:

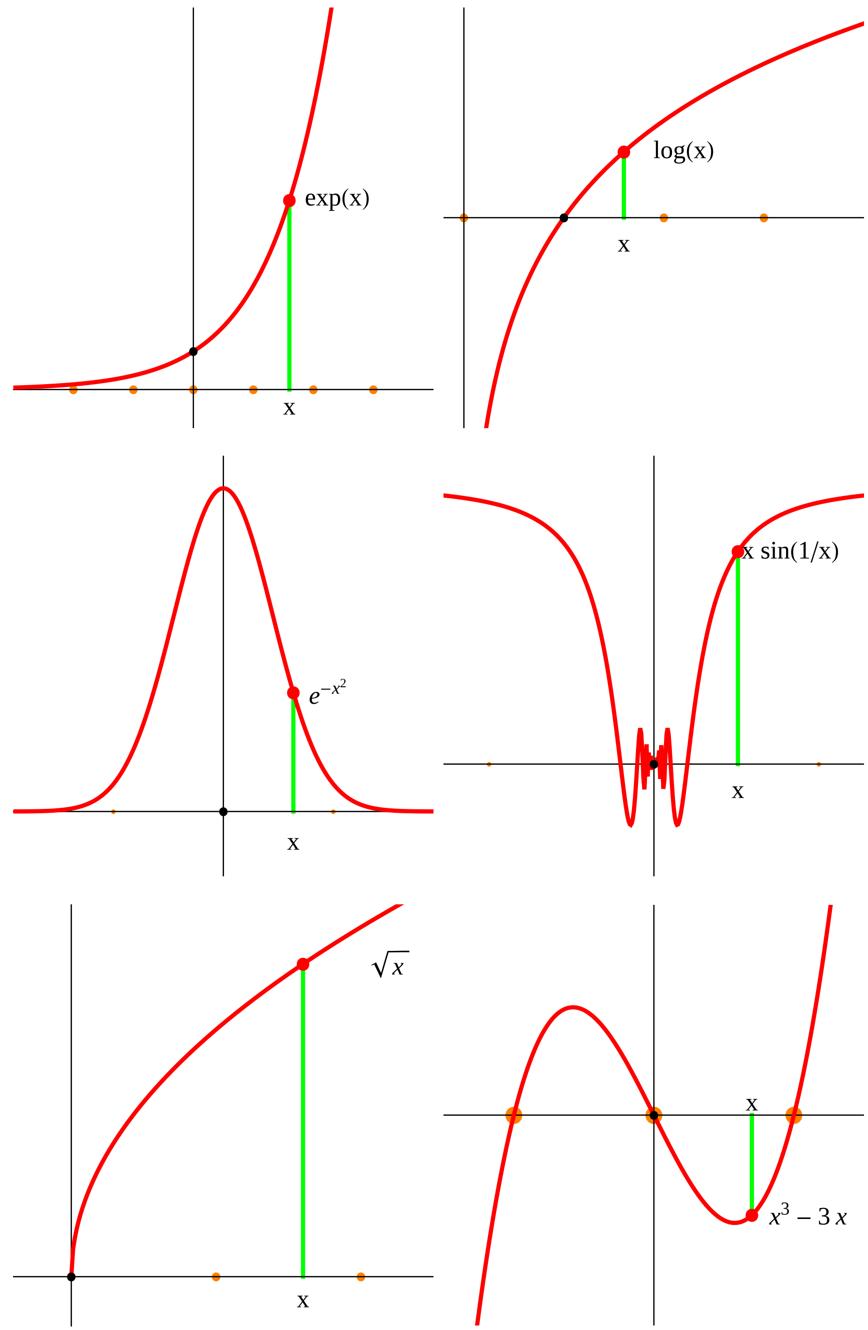
addition	$f(x) + g(x)$
scaling	$2f(x)$
translate	$f(x + 1)$
compose	$f(g(x))$
invert	$f^{-1}(x)$
difference	$f(x + 1) - f(x)$
sum up	$f(x) + f(x + 1) + \dots$

Here are important functions:

polynomials	$x^2 + 3x + 5$
rational functions	$(x + 1)/(x^4 + 1)$
exponential	e^x
logarithm	$\log(x)$
trig functions	$\sin(x)$, $\tan(x)$
inverse trig functions	$\arcsin^{-1}(x), \arctan(x)$.
roots	$\sqrt{x}, x^{1/3}$

We will look at these functions **a lot** during this course. The logarithm, exponential and trigonometric functions are especially important.
For some functions, we need to restrict the domain, where the function is defined. For the square root function $\sqrt{x}$ or the logarithm $\log(x)$ for example, we have to assume that the number is positive. We write that the domain is $(0, \infty) = \mathbf{R}^+$. For the function $f(x) = 1/x$, we have to assume that x is different from zero. Keep these three examples in mind.

The **graph** of a function is the set of points $\{(x, y) = (x, f(x))\ \}$ in the plane, where x runs over the domain A of f. Graphs allow us to **visualize** functions. We can "see them", when we draw the graph.



Homework

1 Draw the function $f(x) = x\sin(x)$. Its graph goes through the origin $(0,0)$.
a) A function is called **odd** if $f(-x) = -f(x)$. Is f odd?
b) A function is called **even** if $f(x) = f(-x)$. Is f even?
c) A function is called **monotone increasing** if $f(y) > f(x)$ if $y > x$. Is f monotone increasing? No need to decide this yet analytically. Just draw(*) and decide.

2 A function $f : A \to B$ is called **invertible** or **one to one** if there is an other function g such that $g(f(x)) = x$ for all x in A and $f(g(y)) = y$ for all $y \in B$. For example, the function $g(x) = \sqrt{x}$ is the inverse of $f(x) = x^2$ as a function from $A = [0,\infty)$ to $B = [0,\infty)$. Determine from the following functions whether they are invertible. If they are invertible, find the inverse.
a) $f(x) = \cos(x)$ from $A = [0,\pi/2]$ to $B = [0,1]$
b) $f(x) = x^5$ from $A = \mathbf{R}$ to $B = \mathbf{R}$
c) $f(x) = x^4$ from $A = \mathbf{R}$ to $B = \mathbf{R}$
d) $f(x) = \exp(2x)$ from $A = \mathbf{R}$ to $B = R^+ = (0,\infty)$.
e) $f(x) = 1/(1+x^2)$ from $A = [0,\infty)$ to $B = [0,\infty)$.

3 Look at the function $f_1(x) = 7x(1-x)/2$, $f_2(x) = f_1(f_1(x))$, $f_3(x) = f_2(f_2(x))$.
a) Draw the graphs of the functions f_1, f_2, f_3 on the interval $[0,1]$.
b) Can you imagine what $f_{100}(x)$ looks like? You might want to make more experiments here to see the answer. Of course you are allowed to plot the functions with a calculator with Wolfram alpha or with Mathematica.

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Plot[NestList[(7/2) (# (1 - #)) &, x, 3], {x, 0, 1}]
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4 Lets call a function $f(x)$ a **composition square root** of a function g if $f(f(x)) = g(x)$. For example, the function $f(x) = x^2+1$ is the composition square root of $g(x) = x^4+2x^2+2$ because $f(f(x)) = (x^2+1)^2+1 = g(x)$. Find the composition square roots of the following functions:
a) $f(x) = \cos(\cos(x))$.
b) $f(x) = x^4$
c) $f(x) = x$
d) $f(x) = x^4 + 2x^2 + 2$
e) $f(x) = e^{e^x}$.

Note that it can be difficult in general to find the square root function in general. Already for basic functions like $\exp(x)$ or $\sin(x)$, we are speechless.

5 A function $f(x)$ has a **root** at $x = a$ if $f(a) = 0$. Roots are places, where the function is zero. Find one root for each of the following functions or state that there is none.
a) $f(x) = \cos(x)$
b) $f(x) = \exp(-x^2)$
c) $f(x) = x^3 - x$
d) $f(x) = \sin(x) - 1$
e) $f(x) = \csc(x) = 1/\sin(x)$

(*) Here is how you can use the Web to plot a function. The example given is $\sin(x)$.

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http://www.wolframalpha.com/input/?i=Plot+sin(x)
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