# Lecture 29: Integration by parts

If we integrate the product rule $(uv)' = u'v + uv'$ we obtain an integration rule called **integration by parts**. It is a powerful tool, which complements substitution. As a rule of thumb, always try first to simplify a function and integrate directly, then give substitution a first shot before trying integration by parts.

$$\int u(x)\, v'(x) dx = u(x)v(x) - \int u'(x)v(x)\, dx .$$

1 Find $\int x \sin(x)\, dx$. **Solution.** Lets identify the part which we want to differentiate and call it $u$ and the part to integrate and call it $v'$. The integration by parts method now proceeds by writing down $uv$ and subtracting a new integral which integrates $u'v$ :

$$\int x \sin(x)\, dx = x\,(-\cos(x)) - \int 1\,(-\cos(x))\, dx = -x\cos(x) + \sin(x) + C .$$

2 Find $\int xe^x\, dx$. **Solution.**

$$\int x \exp(x)\, dx = x\exp(x) - \int 1 \exp(x)\, dx = x\exp(x) - \exp(x) + C .$$

3 Find $\int \log(x)\, dx$. **Solution.** There is only one function here, but we can look at it as $\log(x) \cdot 1$

$$\int \log(x)\, 1\, dx = \log(x)x - \int \frac{1}{x} x\, dx = x\log(x) - x + C .$$

4 Find $\int x\log(x)\, dx$. **Solution.** Since we know from the previous problem how to integrate log we could proceed like this. We would get through but what if we do not know? Lets differentiate $\log(x)$ and integrate $x$:

$$\int \log(x)\, x\, dx = \log(x)\frac{x^2}{2} - \int \frac{1}{x}\frac{x^2}{2}\, dx$$

which is $\log(x)x^2/2 - x^2/4$.

We see that it is better to differentiate log first.

5 **Marry go round:** Find $I = \int \sin(x)\exp(x)\, dx$. **Solution.** Lets integrate $\exp(x)$ and differentiate $\sin(x)$.

$$= \sin(x)\exp(x) - \int \cos(x)\exp(x)\, dx .$$

Lets do it again:

$$= \sin(x)\exp(x) - \cos(x)\exp(x) - \int \sin(x)\exp(x)\, dx.$$

We moved in circles and are stuck! Are we really. We have derived an identity

$$I = \sin(x)\exp(x) - \cos(x)\exp(x) - I$$

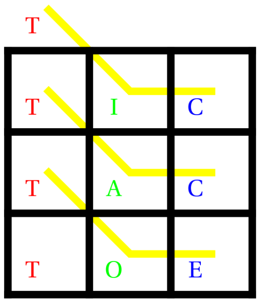
which we can solve for $I$ and get

$$I = [\sin(x)\exp(x) - \cos(x)\exp(x)]/2 .$$





## Tic-Tac-Toe




Integration by parts can bog you down if you do it several times. Keeping the order of the signs can be daunting. This is why a **tabular integration by parts method** is so powerful. It has been called "Tic-Tac-Toe" in the movie Stand and deliver. Lets call it Tic-Tac-Toe therefore.

6 Find the anti-derivative of $x^2\sin(x)$. **Solution:**

| $x^2$ | $\sin(x)$ | |
|---|---|---|
| $2x$ | $-\cos(x)$ | $\oplus$ |
| $2$ | $-\sin(x)$ | $\ominus$ |
| $0$ | $\cos(x)$ | $\oplus$ |

The antiderivative is

$$-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C .$$

7 Find the anti-derivative of $(x-1)^3 e^{2x}$. **Solution:**

| $(x-1)^3$ | $\exp(2x)$ | |
|---|---|---|
| $3(x-1)^2$ | $\exp(2x)/2$ | $\oplus$ |
| $6(x-1)$ | $\exp(2x)/4$ | $\ominus$ |
| $6$ | $\exp(2x)/8$ | $\oplus$ |
| $0$ | $\exp(2x)/16$ | $\ominus$ |

The anti-derivative is

$$(x-1)^3e^{2x}/2 - 3(x-1)^2e^{2x}/4 + 6(x-1)e^{2x}/8 - 6e^{2x}/16 + C .$$

8 Find the anti-derivative of $x^2\cos(x)$. **Solution:**

| $x^2$ | $\cos(x)$ | |
|---|---|---|
| $2x$ | $\sin(x)$ | $\oplus$ |
| $2$ | $-\cos(x)$ | $\ominus$ |
| $0$ | $-\sin(x)$ | $\oplus$ |

The anti-derivative is

$$x^2\sin(x) + 2x\cos(x) - 2\sin(x) + C .$$

Ok, we are now ready for more extreme stuff.

9 Find the anti-derivative of $x^7\cos(x)$. **Solution:**

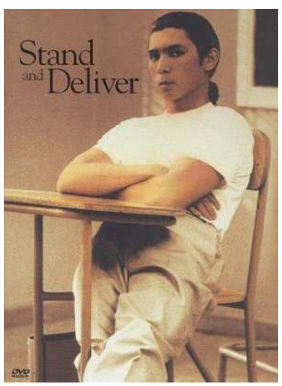
| $x^7$ | $\cos(x)$ | |
|---|---|---|
| $7x^6$ | $\sin(x)$ | $\oplus$ |
| $42x^5$ | $-\cos(x)$ | $\ominus$ |
| $120x^4$ | $-\sin(x)$ | $\oplus$ |
| $840x^3$ | $\cos(x)$ | $\ominus$ |
| $2520x^2$ | $\sin(x)$ | $\oplus$ |
| $5040x$ | $-\cos(x)$ | $\ominus$ |
| $5040$ | $-\sin(x)$ | $\oplus$ |
| 0 | $\cos(x)$ | $\ominus$ |

The anti-derivative is

$$\begin{aligned} F(x) &= x^7\sin(x) \\ &+ 7x^6\cos(x) \\ &- 42x^5\sin(x) \\ &- 210x^4\cos(x) \\ &+ 840x^3\sin(x) \\ &+ 2520x^2\cos(x) \\ &- 5040x\sin(x) \\ &- 5040\cos(x) + C\,. \end{aligned}$$

Do this without this method and you see the value of the method.

[1] [2] [3]:




I myself learned the method from the movie "Stand and Deliver", where **Jaime Escalante** of the Garfield High School in LA uses the method. It can be traced down to an article of V.N. Murty. The method realizes in a clever way an iterated integration by parts method:

$$\int fg\,dx = fg^{(-1)} - f^{(1)}g^{-2} + f^{(2)}g^{(-3)} - \dots - (-1)^n \int f^{(n+1)}g^{(-n-1)}\,dx$$

which can easily shown to be true by induction and justifies the method: the $f$ function is differentiated again and again and the $g$ function is integrated again and again. You see, where the alternating minus signs come from. You see that we always pair a $k$'th derivative with a $k+1$'th integral and take the sign $(-1)^k$.

## Coffee or Tea?

[1]V.N. Murty, Integration by parts, Two-Year College Mathematics Journal 11, 1980, pages 90-94.
[2]David Horowitz, Tabular Integration by Parts, College Mathematics Journal, 21, 1990, pages 307-311.
[3]K.W. Folley, integration by parts, American Mathematical Monthly 54, 1947, pages 542-543



When doing integration by parts, We want to try first to differentiate **L**ogs, **I**nverse trig functions, **P**owers, **T**rig functions and **E**xponentials. This can be remembered as **LIPTE** which is close to "lipton" (the tea).

For coffee lovers, there is an equivalent one: **L**ogs, **I**nverse trig functions, **A**lgebraic functions, **T**rig functions and **E**xponentials which can be remembered as **LIATE** which is close to "latte" (the coffee).

Whether you prefer to remember it as a "coffee latte" or a "lipton tea" is up to you.

There is even a better method, the "opportunistic method":

> Just integrate what you can integrate and differentiate the rest.

An don't forget to consider integrating 1, if nothing else works.

LIATE

LIPTE

## Homework

1 Integrate $\int x^2\log(x)\,dx$.
2 Integrate $\int x^5\sin(x)\,dx$
3 Find the anti derivative of $\int x^6\exp(x)\,dx$. (*)
4 Find the anti derivative of $\int \sqrt{x}\log(x)\,dx$.
5 Find the anti derivative of $\int \sin(x)\exp(-x)\,dx$.

(*) If you want to go for the record. Lets see who can integrate the largest $x^n\exp(x)$! It has to be done by hand, not with a computer algebra system although.