Lecture 5: Intermediate Value Theorem

If $f(a) = 0$, then the value a is called a **root** of f. For example, $f(x) = \cos(x)$ has the root $x = \pi/2$.

1 $f(x) = 4x + 6$. Find the roots of f. **Answer**: set the function equal to 0 and solve for x. We get $4x + 6 = 0$

2 $f(x) = x^2+2x+1$ Find the roots of f. **Answer**: we can write $f(x) = (x+1)^2$. The function has the root $x = -1$.

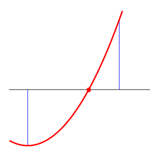
3 $f(x) = (x - 2)(x + 6)(x + 3)$. Find the roots of f.

4 $f(x) = 12 + x - 13x^2 - x^3 + x^4$. Find the roots of f. We do not have a formula for this, but we can try. Indeed, we see that for $x = 1, x = -3, x = 4, x = -1$ we have roots.

5 $f(x) = \exp(x)$. This function does not have any root.

6 $f(x) = 2^x - 16$ has the root $x = 2$.

Intermediate value theorem of Bolzano. If f is continuous on $[a, b]$ and $f(a), f(b)$ have different signs, there is a root of f in (a, b).



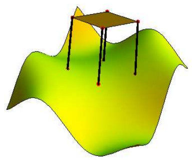
Proof. We can assume $f(a) < 0$ and $f(b) > 0$. The other case is similar. Look at the point $c = (a + b)/2$. If $f(c) < 0$, then look take $[c, b]$ as your new interval, otherwise, take $[a, c]$. We get a new root problem on a smaller interval. Repeat the procedure. After n steps, the search is narrowed to an interval $[u_n, v_n]$ of size $2^{-n}(b - a)$. Continuity assures that $f(u_n) - f(v_n) \to 0$ and $f(u_n), f(v_n)$ have different signs. Both u_n, v_n converge to a root of f.

7 The function $f(x) = x^{17} - x^3 + x^5 + 5x^7 + \sin(x)$ has a root. Solution. The function goes to $+\infty$ for $x \to \infty$ and to $-\infty$ for $x \to -\infty$. We have for example $f(10000) > 0$ and $f(-1000000) < 0$. The intermediate value theorem assures there is a point where $f(x) = 0$.

8 There is a solution to the equation $x^x = 10$. Solution: for $x = 1$ we have $x^x = 1$ for $x = 10$ we have $x^x = 10^{10} > 10$. Apply the intermediate value theorem.

9 There exists a point on the earth, where the temperature is the same as the temperature on its antipode. Solution: Lets draw a meridian through the north and south pole and let $f(x)$ be the temperature on that circle. Define $g(x) = f(x) - f(x + \pi)$. If this function is zero on the north pole, we have found our point. If not, $g(x)$ different signs on the north and south pole. There exists therefore a point, where the temperature is the same.

10 **Wobbly Table Theorem.** On an arbitrary floor, a square table can be turned so that it does not wobble any more.



Why? The 4 legs ABCD are on a square. Let x be the angle of the line AC with with some coordinate axes if we look from above. Given the angle x, we can position the table **uniquely** as follows: the center of ABCD is on the z-axes, the legs ABC are on the floor and AC points in the direction x. Let $f(x)$ denote the height of the fourth leg D from the ground. If we find an angle x such that $f(x) = 0$, we have a position where all four legs are on the ground. Assume $f(0)$ is positive. (If it is negative, the argument is similar.) Tilt the table around the line AC so that the two legs B,D have the same vertical distance h from the ground. Now translate the table down by h. This does not change the angle x nor the center of the table. The two previously hovering legs BD now touch the ground and the two others AC are below. Now rotate around BD so that the third leg C is on the ground. The rotations and lowering procedures have not changed the location of the center of the table nor the direction. This position is the same as if we had turned the table by $\pi/2$. Therefore $f(\pi/2) < 0$. The intermediate value theorem assures that f has a root between 0 and $\pi/2$.

Define $Df(x) = (f(x+h)-f(x))/h$. Lets call it the **derivative** of f for the constant h. We will study it more in the next lecture. But you have verified for example $D\exp_h(x) = \exp_h(x)$ in a homework.

Lets call a point p, where $Df(x) = 0$ a **critical point** for h. Lets call a point a a **local maximum** if $f(a) \geq f(x)$ in an open interval containing a. Define similarly a **local minimum** as a point where $f(a) \leq f(x)$.

11 The function $f(x) = x(x - h)(x - 2h)$ has the derivative $Df(x) = 3x(x - h)$ as you have verified in the case $h = -1$ in the first lecture of this course in a worksheet. We will write $[x]^3 = x(x - h)(x - 2h)$ and $[x]^2 = x(x - h)$. The computation just done tells that $D[x]^3 = 3[x]^2$. Since $[x]^2$ has exactly two roots $0, h$, the function $[x]^3$ has exactly 2 critical points.

12 More generally for $[x]^{n+1} = x(x - h)(x - 2h)...(x - nh)$ we have $D[x]^{n+1} = (n + 1)D[x]^n$. Because $[x]^n$ has exactly n roots, the function $[x]^{n+1}$ has exactly n critical points. Keep the formula

$$\boxed{D[x]^n = n[x]^{n-1}}$$

in mind!

13 The function $\exp_h(x) = (1 + h)^{x/h}$ satisfies $D\exp_h(x) = \exp_h(x)$. Because this function has no roots and the derivative is the function itself, the function has no critical points. Indeed, this function is monotone.

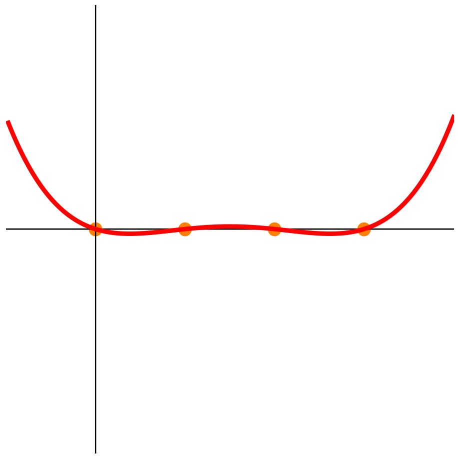


Figure: We see the function $[x]^4 = x(x-h)(x-2h)(x-3h)$ with $h = 0.5$. This function has 3 critical points because $D[x]^4 = 4[x]^3$ and $[x]^3$ has roots at $0, h, 2h$. There are three local maxima or minima according to the theorem.
Later in the course, we will look at the derivative Df in the limit when $h \to 0$. And then the critical points are places where the tangent is horizontal. In our case now, a critical point is a point so that if we walk by a step h to the right, the function does not change. For now, just remember the formula $D[x]^n = n[x]^{n-1}$. It will be the same formula later on when we go to the limit $h \to 0$.

Critical points lead to extrema as we will see later in the course. In our discrete setting we can say:

Fermat's maximum theorem If f is continuous and has a critical point a for h, then f has either a local maximum or local minimum inside the open interval $(a, a+h)$.

Look at the range of the function f restricted to $[a, a+h]$. It is a bounded interval $[c, d]$ by the intermediate value theorem. There exists especially a point u for which $f(u) = c$ and a point v for which $f(v) = d$. These points are different if f is not constant on $[a, a+h]$. There is therefore one point, where the value is different than $f(a)$. If it is larger, we have a local maximum. If it is smaller we have a local minimum.

14 **Problem.** Verify that a cubic polynomial has maximally 2 critical points. **Solution** $f(x) = ax^3 + bx^2 + cx + d$. Because the x^3 terms cancel in $f(x+h) - f(x)$, this is a quadratic polynomial. It has maximally 2 roots.

Homework

1 Find the roots for $f(x) = -30 + 49x - 19x^2 - x^3 + x^4$

2 Use the intermediate value theorem to find a root of $f(x) = x^2 - 6x + 8$ on $[0, 3]$. Are all roots in this interval?

3 a) Argue why there was a time, when Lady Gaga's height was exactly 1 meter and not one mm more less.
b) And that there was a time, when she weighed 50 kg and not a milligram more or less.
c) Was there a time, when she owned exactly 1'000'000 dollars and not one dime more or less?

4 Argue why there is a solution to
a) $\cos(x) = x$.
b) $\exp(x) = x$.
c) $\text{sinc}(x) = x^4$.

5 a) Draw the graph of $f(x) = x^3 - x$.
b) Locate the local maxima and minima.
c) Find the critical points of f to the constant $h = 1$. That means, find the places, where $f(x+1) - f(x) = 0$.
d) For every point a you have found in c), verify that there is a local maximum or minimum in $[a, a+1]$.