Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

4/3/2024: Second Hourly Practice B

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

Please write neatly. Use the same page for the answer if possible.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		100

Problem 1) TF questions (10 points) No justifications are needed.

1) T F The chain rule assures that $\int e^{\sin(x)} \cos(x)\, dx = e^{\sin(x)} + C$

2) T F A Riemann sum approximation S_n of $S = \int_0^1 x^5\, dx$ satisfies $|S - S_n| \leq 5/n$.

3) T F We have $\int_0^1 10 f(x)\, dx = 10 \int_0^1 f(x)\, dx$.

4) T F $\int_0^1 \frac{1}{f(x)}\, dx = \ln(f(x))$.

5) T F If $f(x) = 0$ everywhere, then $\int_0^1 f(x)\, dx = C$

6) T F If $f \leq g$ then $\int_a^b g(x) - f(x)\, dx = \int_a^b g(x)\, dx - \int_a^b f(x)\, dx$.

7) T F $\int_a^b \frac{d}{dx} f(x)\, dx = \frac{d}{dx} \int_a^b f(x)\, dx$.

8) T F If a parameter c defining a function changes, then it is possible that the location of the smallest equilibrium changes discontinuously.

9) T F If T is a Newton step, then $T(x) = x - f'(x)/f(x)$.

10) T F If f is a continuous function such that $f(-1) = f(1) = 1$ and $f(0) = -1$, then f has at least two different roots.

Problem 2) Theorems (10 points) No justifications are needed.

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a) Formulate the **fundamental theorem of calculus** for definite integrals.

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b) Formulate the **chain rule**.

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c) Formulate the **intermediate value theorem**.

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d) Formulate the **mean value theorem**.

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e) Formulate **Rolle's theorem**.

Problem 3) Matching (10 points)

a) 5 (points) Which of the following integrals are zero?

Integral	Check if zero
$\int_{-2\pi}^{2\pi} \sin(x)\ dx$	
$\int_{-2\pi}^{2\pi} e^x - e^{-x}\ dx$	
$\int_{-2\pi}^{2\pi} \frac{1}{1+x^2}\ dx$	
$\int_{-2\pi}^{2\pi} x^2 - \frac{1}{x^2}\ dx$	
$\int_{-2\pi}^{2\pi} \|\sin(x)\|\ dx\ dx$	

b) (5 points) For which of the following functions can you apply the mean value theorem MVT for all $[a, b]$.

Function	Check if MVT is applicable
$\|\cos(x)\|$	
e^x	
$1/x$	
$\arctan(x)$	
$\tan(x)$	

Problem 4) Chain rule (10 points)

Find the derivatives of the following functions:

a) $e^{2e^{3x}}$

b) $\ln(2\ln(3x))$

c) $\cos(2\sin(3x))$

d) $\arctan(2e^{3x})$

e) $\tan(2\tan(3x))$

Problem 5) Related rates (10 points)

Assume you are on a roller coaster $y = 2\pi x + \sin(xy)$ and $x' = 3$. What is y' at the point $x = 1$, $y = \pi$?

Problem 6) Implicit Differentiation (10 points)

Assume that y is a function of x and $x^4 + y^4 = x^2 + y^2 - y + 1$. We can not solve the equation for y and differentiate. But we can compute the derivative y' anyway. Do this at $x = 1, y = 1$.

Problem 7) Definite integrals (10 points)

Evaluate the following definite integrals.

a) $\int_1^2 \frac{1}{x} + \frac{1}{x^2} \; dx$

b) $\int_1^2 \frac{3}{1+x^2} \; dx$

c) $\int_1^2 3\sin(5x) \; dx$

d) $\int_1^2 e^{4x} \; dx$

e) $\int_1^2 \tan(x) dx$

Problem 8) Anti derivatives (10 points)

Solve the indefinite integrals.

a) $\int \frac{1}{1-x^2}\, dx$
Hint: write this as $[1/(1-x)+1/(1+x)]/2$.

b) $\int \frac{\sin(x)}{\cos^2(x)}\, dx$

c) $\int \cot(x)\, dx$

d) $\int x^5 - \frac{1}{x^5}\, dx$

e) $\int \frac{2x}{1+x^2}\, dx$

Problem 9) Newton Step (10 points)

a) Do a Newton step $T(x) = x - f(x)/f'(x)$ to find the root of $\cos(\pi x) - x$ starting at $x_0 = 0$.

b) Now do a second Newton step.

Problem 10) Catastrophes (10 points)

We look at the one-parameter family of functions $f_c(x) = 2x^3 + cx^2$, where c is a parameter.

a) (2 points) Find the critical points of $f_3(x)$.

b) (2 points) Find the critical points of $f_{-3}(x)$.

c) (2 points) Check that 0 is always a critical point.

d) (2 points) For which c is 0 a minimum?

e) (2 points) For which c does the catastrophe occur?