Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

4/3/2024: Second Hourly Practice A

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

Please write neatly. Use the same page for the answer if possible.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		100

Problem 1) TF questions (10 points) No justifications are needed.

1) T F Differentiating $f(f^{-1}(x)) = x$ allows to get $\frac{d}{dx} f^{-1}(x)$.

Solution:
By implicit differentiation.

2) T F $\int_0^1 x^3 \, dx = \lim_{n\to\infty} \sum_{k=0}^{n-1} (\frac{k}{n})^3 \frac{1}{n}$.

Solution:
By definition of the integral.

3) T F If $0 \leq f(x) \leq 1$, then $0 \leq \int_0^1 f(x) \, dx \leq 1$

Solution:
Think about the area. It is in a square of area 1.

4) T F If $f(x)$ is differentiable integral $\int_0^1 f(x) \, dx$ can be approximated by Riemann sums.

Solution:
We have defined it as such.

5) T F If $f(x) = 1$ everywhere, then $\int_a^b f(x) \, dx$ is the length of the interval $[a, b]$.

Solution:
Indeed the integral is $b - a$ then. Since we wrote $[a, b]$ this indicates that $a \leq b$.

6) T F

If f is continuous, then $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$.

Solution:
Yes, we can take the $-$ sign outside the integral.

7) T F

The fundamental theorem of calculus implies $\int_a^b f''(x)\,dx = f'(b) - f'(a)$ if f'' is a differentiable function.

8) T F

The family $f_c(x) = x^2 + c$ experiences a catastrophe at $c = 0$.

Solution:
The nature of the critical point does not change when deforming c.

9) T F

If f is differentiable and T is a Newton step, then $T(x), T^2(x) \dots$ converges to a root of x.

Solution:
The root might not exist. Like for $1/x$ we saw that $T(x) = 2x$.

10) T F

If f is continuous, then $\int_a^b f(-x)\,dx = -\int_a^b f(x)\,dx$.

Solution:
This is not true in general. It would be true for odd functions.

Problem 2) Theorems (10 points) No justifications are needed.

Fill in the missing part into the empty box to make a true statement.

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a) $\int_0^1 f'(x)\,dx =$ [ ] by the **fundamental theorem of calculus**.

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b) $\int_2^5$ [ ] $dt = f(5) - f(2)$ by the **fundamental theorem of calculus**.

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c) A continuous function f for which $f(-1) = -3$ and $f(1) = 8$ has a [ ] by the [ ] theorem.

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d) If a differentiable function satisfies $f(a) = f(b)$ then there is a point for $f'(x) = 0$ by the [ ] theorem which is a special case of the mean value theorem.

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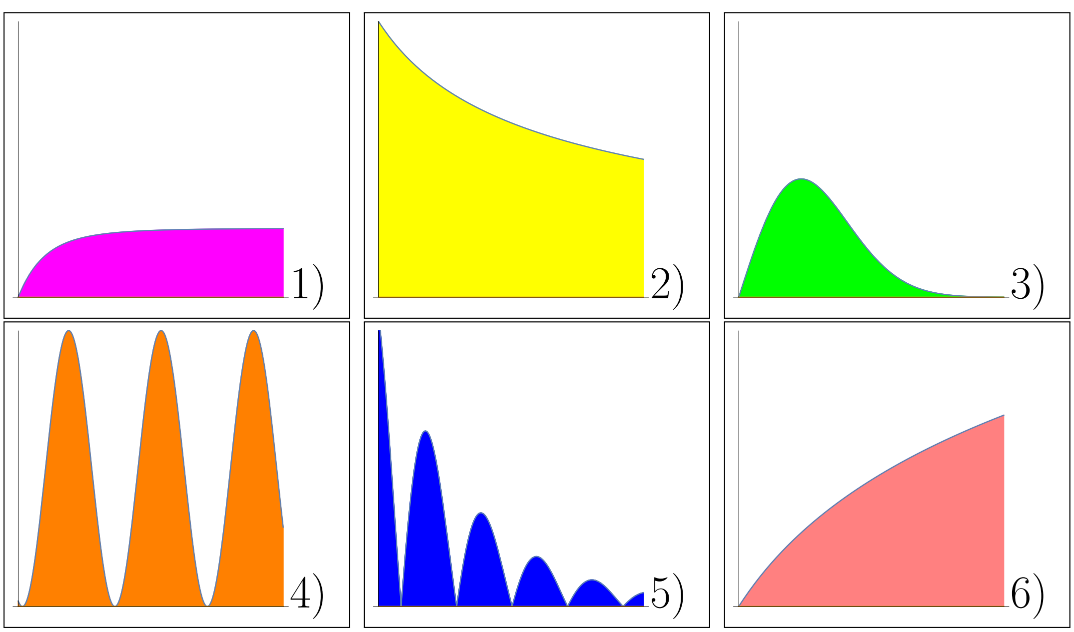
e) Assume $f_c(x)$ is a **family of functions** such that for $c < 0$, there is no minimum and for $c > 0$ there is one minimum, then c is called a [ ].

Solution:
a) $f(1) - f(0)$
b) $f'(x)$
c) Root
d) Rolle
e) Catastrophe

Problem 3) Matching (10 points)

Match the following integrals with parts of the regions.

Integral	Fill in 1-6
$\int_1^5 \sin^2(3x)\ dx$	
$\int_1^5 \frac{1}{x^{1/2}}\ dx$	
$\int_1^5 (x-1)e^{-(x-1)^2}\ dx$	
$\int_1^5 \log(x)\ dx$	
$\int_1^5 \frac{x^4-1}{x^4+1}\ dx$	
$\int_1^5 3\|\sin(5x)\|e^{-x}\ dx$	



Solution:
4,2,3,6,1,5.

Problem 4) Chain rule (10 points)

Compute the following derivatives:

a) $5\tan(x^2)$

b) $\cos(\sin(3x)) + e^{\cos(x)}$

c) $\cos^7(x) + \cos^3(8x)$

d) $3\sqrt{x^2 + x^5}$

e) $8\ln(2\ln(x^3))$

Solution:
a) $10x/\cos^2(x^2)$
b) $-\sin(\sin(3x))3\cos(3x) - e^{\cos(x)}\sin(x)$
c) $7\cos^6(x)(-\sin(x)) + 3\cos^2(8x)(-\sin(8x)8)$
d) $(3/2)(x^2 + x^5)^{-(1/2)}(2x + 5x^4)$
e) $(8/(2\ln(x^3)))(2/x^3)3x^2 = \frac{24}{x\ln(x^3)}$

Problem 5) Related rates (10 points)

A container of length 10, width $2z$ at height z contains water of volume

$$V(z) = 10z^2 \ .$$

If the volume $V(z(t))$ decreases with constant rate $V' = -1$, how fast does the water level $z(t)$ sink when $t = 1, V = 10$?

Solution:
Differentiate the relation. $-1 = V' = 20zz'$ shows that $z'(t) = -1/(20z(t))$. At $t = 1$ we have $z = 1$ and $-1/(20z(1))$ which is $-1/20$.

Problem 6) Implicit Differentiation (10 points)

You know

$$y^5x^3 + \cos(x-2) + y^3 + x = 12$$

and that y is a function of x. Assuming, $x = 2, y = 1$, what is $y'(2)$?

Solution:
Differentate the relation.

$$5y^4y'x^3 + y^5 3x^2 - \sin(x-2) + 3y^2y' + 1 = 0 \ .$$

Now fill in the numbers $x = 2, y = 1$ to get

$$40y' + 3 * 4 - 0 + 3y' + 1 = 0$$

and so $y' = -13/43$.

Problem 7) Definite integrals (10 points)

Evaluate the following definite integrals.

a) $\int_0^1 \frac{6}{(x+3)^5} \, dx$

b) $\int_0^1 3x^3/(1+x^4) \, dx$

c) $\int_0^1 17/(1+x^2) \, dx$

d) $\int_0^e \ln(e+x)/(e+x) \, dx$

e) $\int_0^1 99e^{22x} + 5x^7 \, dx$

Solution:

a) $\frac{-6}{4(x+3)^4}|_0^1 = (3/2)(1/4^4 - 1/3^4) = -\frac{175}{13824}$.

b) $(3/4)\ln(1+x^4)|_0^1 = (3/4)\ln(2)$

c)$17\arctan(x)|_0^1 = 17(\arctan(1) - \arctan(0)) = 17\pi/4$.

d)$\ln^2(e+x)/2|_0^e = (\ln^2(2e) - \ln^2(e))/2$

e) $(99/22)e^{22x} + 5x^8/8|_0^1 = (99/22)(e^{22} - 1) + 5/8$

Problem 8) Anti derivatives (10 points)

Solve the indefinite integrals.

a) $\int 4x^9 + \frac{3}{x}\, dx$

b) $\int \cos(x) + \sin(x) + \tan(x)\, dx$

c) $\int \cos^2(2x) + \sin^2(3x)\, dx$

d) $\int x^6 e^{x^7}\, dx$

e) $\int 3x^2/(1 + x^3)\, dx$

Solution:
a) $(4/10)x^{10} + 3\ln(x) + C$
b) $\sin(x) - \cos(x) - \ln(\cos(x)) + C$.
c) Write $\cos^2(2x) = [1 + 2\cos(4x)]/2$ and $\sin^2(3x) = [1 - 2\cos(6x)]/2$ (double angle formulas). Now we can integrate $x/2 + \sin(4x)/4 + x/2 - \sin(6x)/6 + C$.
d) $e^{x^7}/7 + C$.
e) $\ln(1 + x^3) + C$.

Problem 9) Newton Step (10 points)

a) Do a Newton step $T(x) = x - f(x)/f'(x)$ to find the root of $x^7 - x^3 - x = 0$ starting at $x_0 = 1$.

b) Now do a second Newton step.

Solution:
a) Compute $f'(x) = 7x^6 - 3x^2 - 1$.
So $T(x) = x-(x^7-x^3-x)/(7x^6-3x^2-1)$. Pluggin in $x_0 = 1$ gives $1+1/3 = 4/3$. b) Now plug in $T(4/3) = 4/3 - ((4/3)^7 - (4/3)^3 - 4/3)/(7(4/3)^6 - 3(4/3)^2 - 1))$ (does not have to be simplified) to get 29312/2405524055.

Problem 10) Catastrophes (10 points)

Consider the family of functions $f(x) = x^3/3+cx$ on the real line.

a) (4 points) Find all critical points of f for $c < 0$ and determine the stable ones or indicate there are none.

b) (4 points) Find all critical points of f for $c > 0$ and determine the stable ones or indicate there are none.

c) (2 points) For which value of c does a catastrophe occur?

Solution:
a) The critical points are $x^2+c = 0$ which means $x = \sqrt{-c}$. There are no critical points for $c > 0$ and two different critical points for $c < 0$.
b) The second derivative is $2x$ which is negative for $x < 0$ and positive for $x > 0$. The point $\sqrt{-c}$ is a local minimum for $c < 0$. There is exactly one critical point which is a minimum for $c < 0$.
c) The catastrophe appears at the parameter $c = 0$ because a local minimum, present for $c < 0$ disappears for $c \geq 0$.