Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

2/28/2024: First Hourly

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

Please write neatly. Use the same page for the answer if possible.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		100

Problem 1) TF questions (10 points) No justifications are needed.

1) T F $\lim_{x \to 0} \frac{\sin(x)}{x} = 0$.

2) T F e^x has a root at $x = 0$.

3) T F $\lim_{x \to 0} \cos(x) = 1$.

4) T F $f(x) = |\sin(x)|$ is differentiable everywhere.

5) T F $f(x) = \frac{1-x^5}{1-x}$ has a limit at $x = 1$.

6) T F The function $f(x) = x^7 + 7$ has a critical point at $x = 0$.

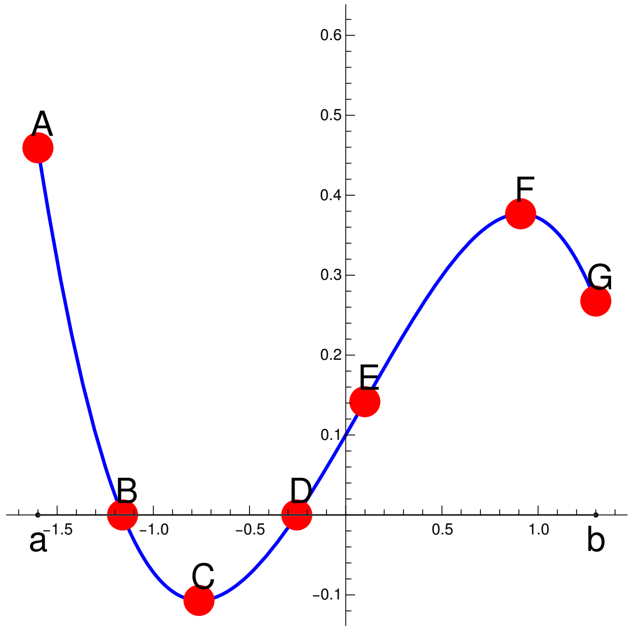
7) T F The function $f(x) = \sin(x)\sin(\frac{1}{x})$ (with the understanding $f(0) = 0$) is continuous everywhere.

8) T F $\frac{d}{dx} \ln|5x| = \frac{5}{x}$.

9) T F $\frac{d}{dx} \ln|7 - 2x| = \frac{-2}{7-2x}$.

10) T F The derivative of $\frac{g}{f}$ is $\frac{fg' - f'g}{f^2}$.

Problem 2) Analysis (10 points) No justifications are needed.



a) (2 points) List the points A-G that are critical points of f.

b) (2 points) List the points A-G that are critical points of f'.

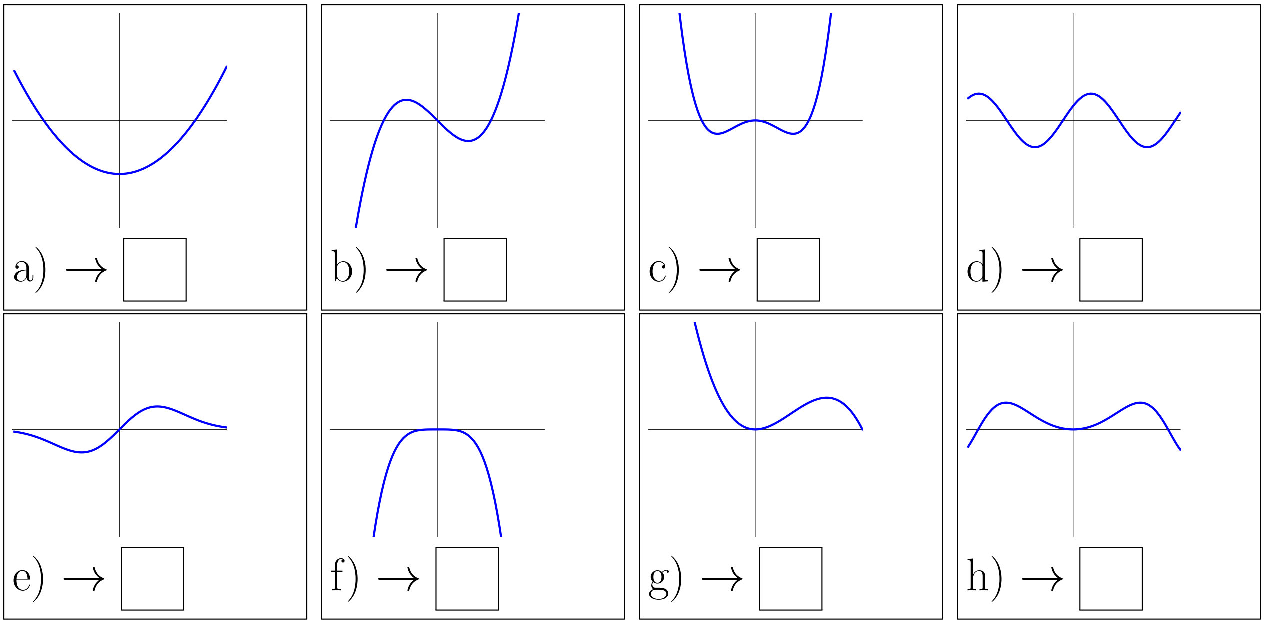
c) (2 points) List the points A-G that are global maxima on the given interval $[a, b]$.

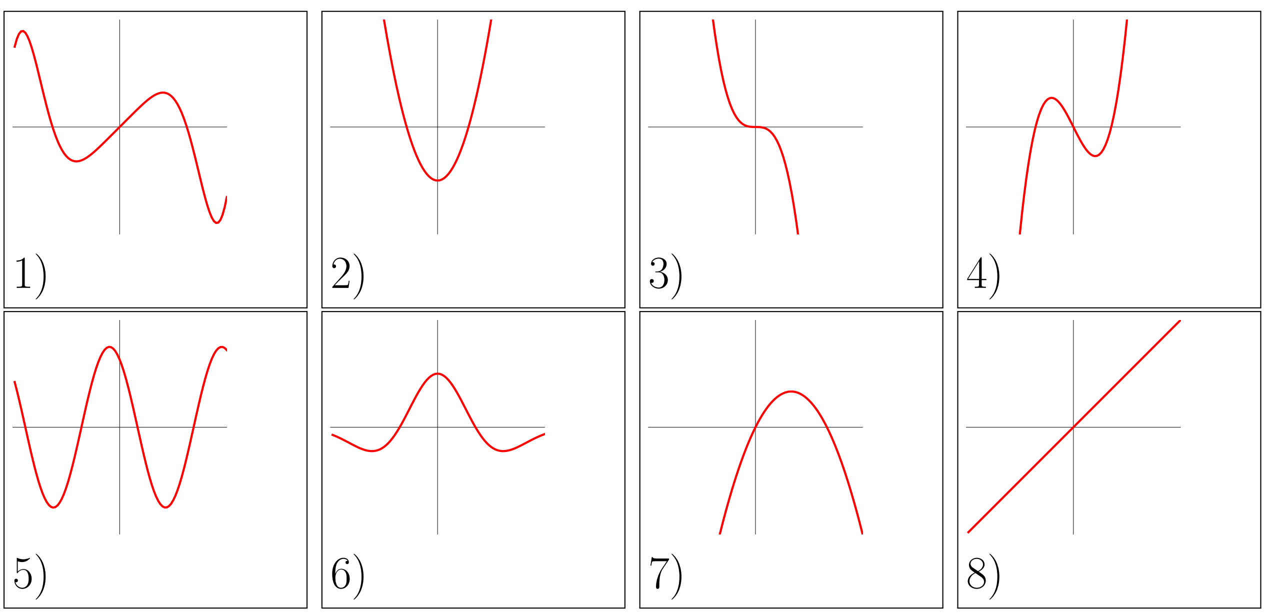
d) (2 points) List the points A-G of f that are global minima on the given interval $[a, b]$.

e) (2 points) Which theorem assures that there is both a global max and global min on $[a, b]$.

Problem 3) Graphing (10 points) No justifications are needed.

The functions f in a) to h) match with their derivatives 1) to 8). Find this correspondence.





Problem 4) Continuity (10 points)

Make the decision "continuous on $[-5,5]$" or "not continuous on $[-5,5]$" in each of the cases a)-e) and point out any possible x values, which need special attention (*) and how you know why we have continuity there.

a) (2 points) $f(x) = \frac{x^2-9}{x-3}$

b) (2 points) $f(x) = \frac{e^{5x}-1}{e^{2x}-1}$.

c) (2 points) $f(x) = \frac{\sin^3(x)}{x^4}$

d) (2 points) $f(x) = ||3x| - 4x|$

e) (2 points) $f(x) = \sin(\sin(\frac{1}{x}))$.

(*) As usual, we extend continuity to functions for which a continuation is possible to initially not defined points, like $f(x) = (x^2 - 1)/(x-1)$, which is considered continuous everywhere because we can fill in a function value for $x = 1$ which makes it continuous.

Problem 5) Derivatives (10 points)

Compute the derivatives. State the differentiation rules used.

a) (2 points) $f(x) = x^2 + 3x + 7$.

b) (2 points) $f(x) = \frac{1}{x} + 3e^x$

c) (2 points) $f(x) = \sin(3x) + \cos(4x)$

d) (2 points) $f(x) = \sin(x)e^x$.

e) (2 points) $f(x) = \frac{\cos(x)}{1+x}$

Problem 6) Limits (10 points)

Find the limits $\lim_{x \to 0} f(x)$ for the following functions. As always, give details and indicate what method you are using.

a) (2 points) $f(x) = \frac{1-e^{8x}}{1-e^{4x}}$.

b) (2 points) $f(x) = \frac{1-x^2}{\cos(3x)}$.

c) (2 points) $f(x) = \frac{3x}{\ln(1+4x)}$.

d) (2 points) $f(x) = \frac{\ln(x^4)}{\ln(x^2)}$.

e) (2 points) $f(x) = x \ln|3x|$.

Problem 7) Functions (10 points) no explanations needed

A function is called **1-periodic** if $f(x+1) = f(x)$ for all x. A function is called **even** if $f(x) = f(-x)$ for all x. A function is called **odd** if $f(x) = -f(-x)$ for all x. A function is **invertible** from $\mathbb{R} \to \mathbb{R}$ if $f(x) = y$ has a unique solution x for every $y \in \mathbb{R}$. Invertible means that you can find an inverse function that works on the entire real line. In the following table, check each box which applies. Do not put anything in a box which does not apply.

Function	1-periodic	odd	even	invertible
$\sin(2\pi x)$				
$\cos(2\pi x)$				
$(x-1)^2$				
x^3				
e^x				

Problem 8) Extrema (10 points)

a) (3 points) Find all the critical points of the function

$$f(x) = x^5 - 5x + 7 .$$

b) (4 points) Classify the critical points using the second derivative test.

c) (3 points) Classify the critical points using the first derivative test.

Problem 9) Algebra (10 points)

Simplify the following expressions. For example, $3^x 3^y$ can be written fewer letters as 3^{x+y} or $\sin^2(x)/\sin(x)$ can be simplified as $\sin(x)$. Your simplified f should be shorter than f. In the second column, fill in the derivative $f'(x)$ of the simplified expression.

	Function f	Simplified f	Derivative f'
a)	$\arccos(\cos(x^2))$		
b)	$2^{(x/\ln(2))}$		
c)	$\frac{\ln(x^4)}{\cos^2(x)+\sin^2(x)}$		
d)	$\cos(3x)\tan(3x)$		
e)	$\frac{(4x)^3}{(2x)^4}$		

Problem 10) Linearization (10 points, 5 points each)

a) (5 points) Use linearization to estimate $996^{1/3}$.

b) (5 points) Use linearization to estimate $1004^{1/3}$.