Math 1A: Introduction to functions and calculus Oliver Knill, Spring 2024

2/28/2024: First Hourly Practice A

"By signing, I affirm my awareness of the standards of the Harvard College Honor Code."

Your Name:

Please write neatly. Use the same page for the answer if possible.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		100

Problem 1) TF questions (10 points) No justifications are needed.

1) T (highlighted) F

$\lim_{x\to 0} \frac{\sin^2(5x)}{(3x)^2} = \frac{5^2}{3^2}$.

Solution:
First factor then use l'Hopital

2) T F (highlighted)

$\ln(|7x|)$ has a root at $x = 0$.

Solution:
The log function has a singularity at $x = 0$.

3) T (highlighted) F

$\cos(\pi/3) = 1/2$.

Solution:
You know that. It is the cosine of 60 degrees. Draw a triangle.

4) T (highlighted) F

$f(x) = |5 + 4\sin(x)|$ is differentiable everywhere.

Solution:
The absolute value does not matter

5) T (highlighted) F

$f(x) = \frac{(x^2-2x+1)^3}{(1-x)^2}$ has a limit at $x = 1$.

Solution:
On the top we have $(x-1)^6$. We can now divide through and get $(1-x)^4$. At $x = 1$ we have now the limit 0.

6) T **F** The function $f(x) = |7x|$ has a critical point at $x = 0$.

Solution:
It is not differentiable at $x = 0$. This is not a critical point.

7) T **F** The function $f(x) = \cos(\frac{1}{7x})$ with the understanding $f(0) = 0$ is continuous everywhere.

Solution:
We can not use the squeeze theorem. There are points arbitrarily close to $x = 0$ where the function is 1 and points arbitrarily close to $x = 0$ where the function is -1.

8) **T** F The function $(\cos(x) - 1)/x^2$ with the understanding $f(0) = -1/2$ is continuous everywhere.

Solution:
Indeed, the limit for $x \to 0$ exists and is $1/2$ by lHospital.

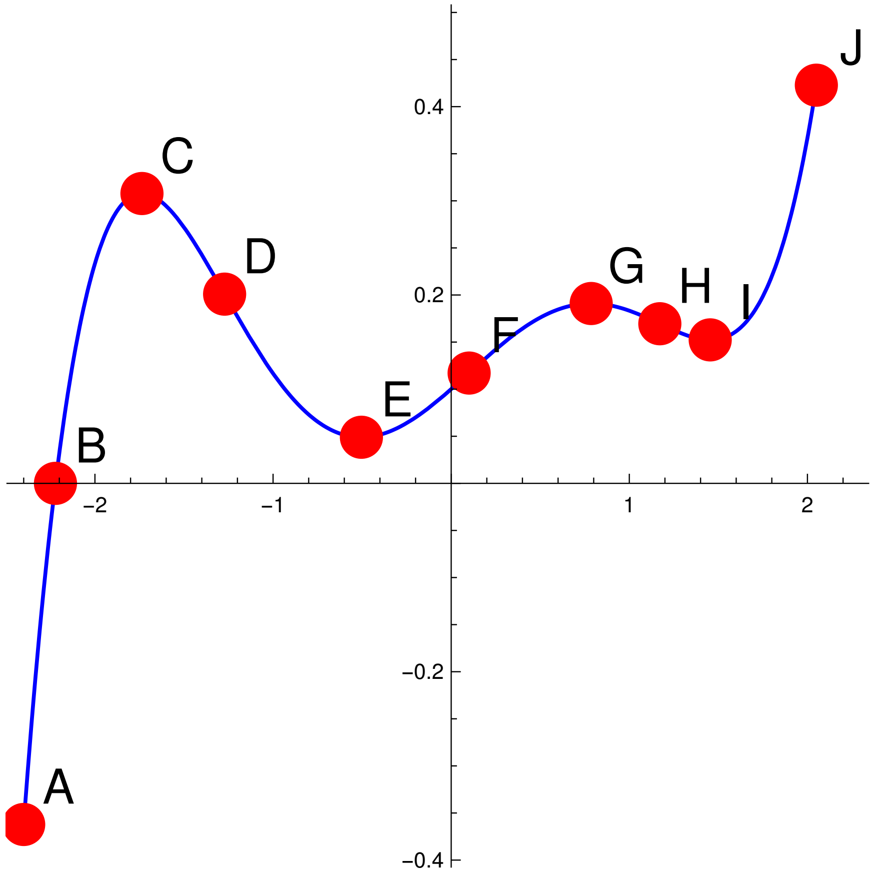
9) **T** F $\frac{d}{dx} \ln(7 - x) = \frac{-1}{7-x}$.

Solution:
There is indeed a negative sign.

10) T **F** The derivative of f/g is $\frac{f'g'-fg}{g^2}$.

Solution:
False formula

Problem 2) Analysis (10 points) No justifications are needed.



a) (2 points) List the points A-J which are roots of f.

b) (2 points) List the points A-J are inflection points.

c) (2 points) List the points A-J that are local maxima.

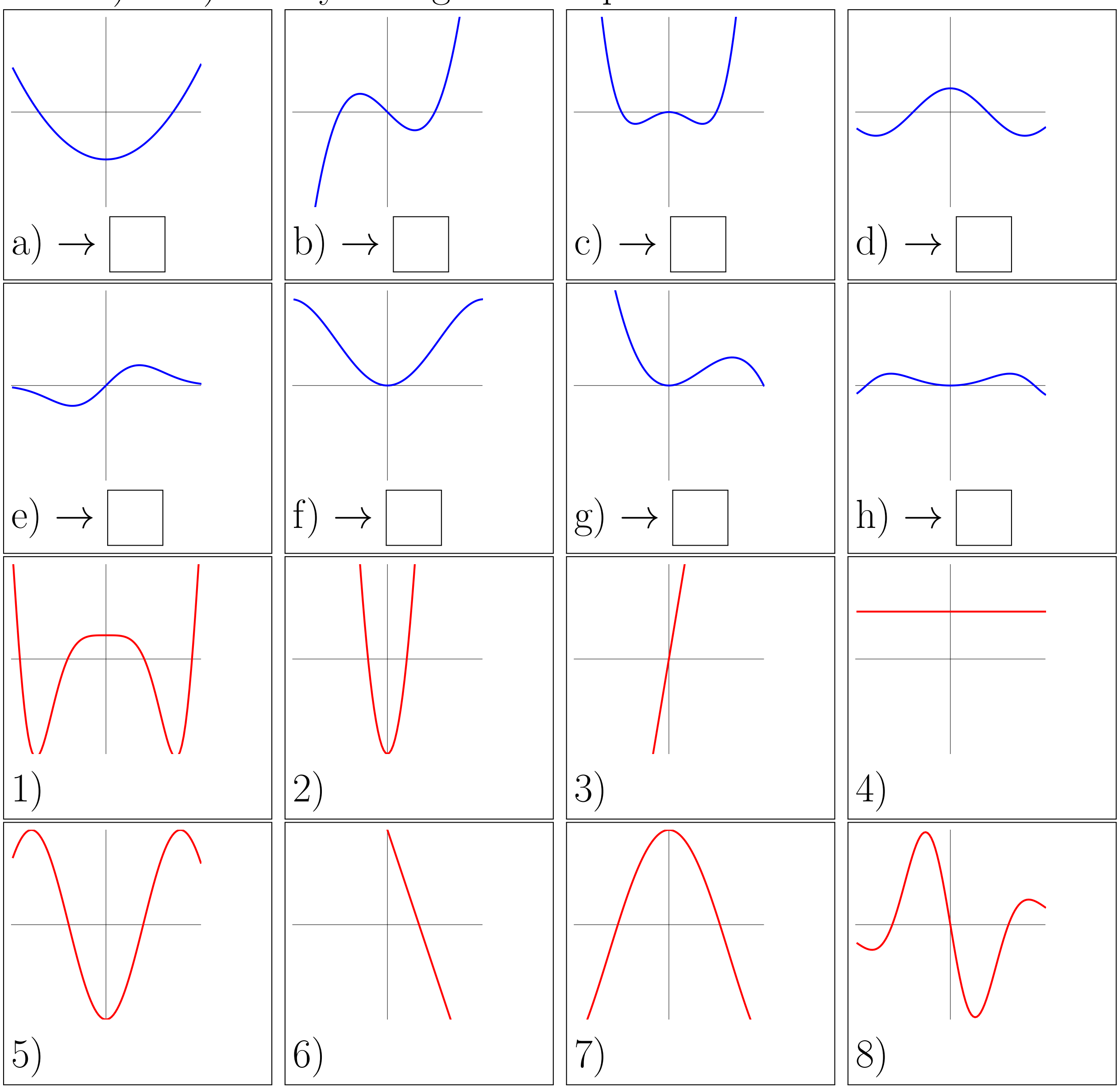
d) (2 points) List the points A-J that are local minima.

e) (2 points) List the points A-J that are global maxima.

Solution:
a) B
b) D, F, H.
c) C, G
d) E, I
e) J.

Problem 3) Graphing (10 points) No justifications are needed.

Match the functions f a) to h) with the second derivatives functions 1) to 8). Every wrong box is a point off.



Solution:
4, 3, 2, 5, and 8, 7, 6, 1

Problem 4) Continuity (10 points)

Which of the following functions are continuous on $[-1, 1]$? As usual we extend the domain of definition to points, where a continuation is possible. In each case make the decision "continuous" or "not continuous" and also point to the x-value or values which need special attention.

a) (2 points) $f(x) = \frac{x^6-1}{x^2-1}$

b) (2 points) $f(x) = \frac{\sin(\sin(x))}{\sin(\sin(\sin(x)))}$.

c) (2 points) $f(x) = \frac{\sin^2(x)}{2+\sin(x^2)}$

d) (2 points) $f(x) = \ln|x|e^x$

e) (2 points) $f(x) = \frac{\sin(\tan(x))}{\sin(x)}$

Solution:

a) Continuous. There is only one point which needs to be checked: x=1 Hospital gives the limit $6/2 = 3$ at $x = 1$.

b) Continuous. There is only point 0 which needs to be considered. The limit at $x = 0$ is 1.

c) Continuous. The denominator is always positive. The function is continuous.

d) Not continuous. There is no way to save this at $x = 0$ as $\ln|x|$ goes to $-\infty$ and $e^0 = 1$.

e) Continuous. We can use Hospital to get the value 1 at $x = 0$ or $x = \pi$ etc.

(*) As usual, we extend continuity to functions for which a continuation is possible to initially not defined points, like $f(x) = (x^2 - 1)/(x - 1)$, which is considered continuous everywhere because we can fill in a function value for $x = 1$ which makes it continuous.

Problem 5) Derivatives (10 points)

Find the derivatives of the following functions. In each case, indicate which differentiation rule you use.

a) (2 points) $f(x) = \frac{1}{1+e^x}$.

b) (2 points) $f(x) = \cos(x)\sin(x)$.

c) (2 points) $f(x) = \frac{1+x^3}{1+x^2}$.

d) (2 points) $f(x) = \arctan(7x) + \sin(3x)$.

e) (2 points) $f(x) = \ln(3x) + \ln(5x))$.

Solution:
a) $-e^x/(1+e^x)^2$.
b) $\cos^2(x) - \sin^2(x)$.
d) $[2x(1+x^2) - 3x^2(1+x^2)]/(1+x^2)^2$.
d) $7(1+(7x)^2) + 3\cos(3x)$
e) $2/x$

Problem 6) Limits (10 points)

Find the limits $\lim_{x\to 0} f(x)$ for each of the following functions:

a) (2 points) $f(x) = \frac{e^{7x}-1}{e^{4x}-1}$.

b) (2 points) $f(x) = \frac{x-1}{x+1}$.

c) (2 points) $f(x) = \frac{\arctan(x)}{\sin(x)}$.

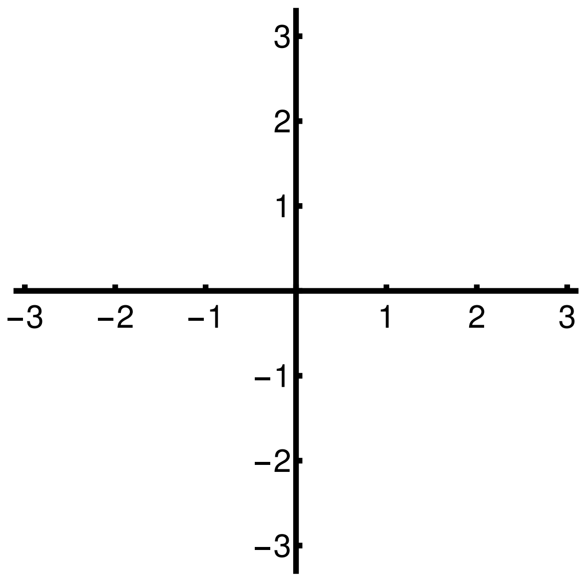
d) (2 points) $f(x) = \frac{\ln(x^3)}{\ln(x^2)}$.

e) (2 points) $f(x) = \frac{\sin(3x)\sin(5x)}{\sin(7x)\sin(2x)}$.

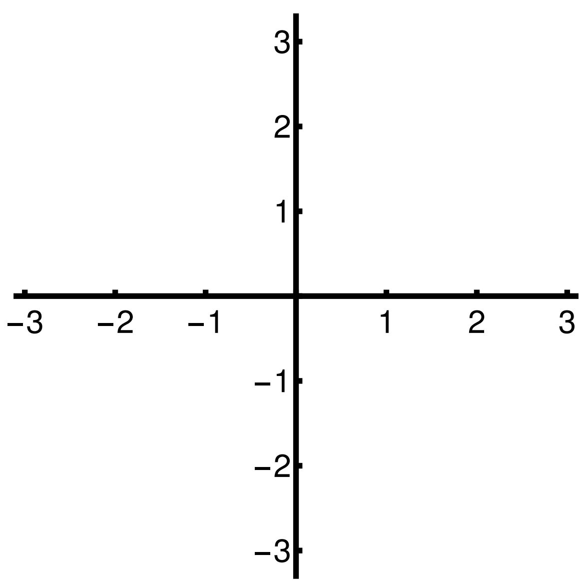
Solution:
All with Hospital.
a) 7/4.
b) -1 (no limit needs to be taken as the values are finite.
c) 1
d) 3/2
e) Take apart: $\sin(3x)/\sin(7x)$ goes to 3/7 and $\sin(5x)/\sin(2x)$ goes to 5/2. We have 15/14.

Problem 7) Functions (10 points) no explanations needed

a) (5 points) Draw the graph of the natural log function $f(x) = \log|x| = \ln|x|$. Make sure you get the roots and asymptotes correct.



b) (5 points) Draw the graph of the arctan function $f(x) = \arctan(x)$. Make sure you get the roots and asymptotes correct.



Problem 8) Extrema (10 points)

a) (3 points) Find all the critical points of the function

$$f(x) = x^3 - 3x + 1 .$$

b) (3 points) Use the second derivative test to classify the critical points of f.

c) (2 points) On the interval $[-3, 3]$, where it the global maximum, and where is the global minimum?

d) (2 points) Which theorem assures that there is a global maximum and a global minimum on $[-3, 3]$?

Solution:
a) We have $f'(x) = 3x^2 - 3$ which is zero at $x = 1$ or $x = -1$.
b) The second derivative is $6x$. At $x = 1$, this is 6, so $x = 1$ is a minimum. At $x = -1$ the second derivative is -6 and $x = -1$ is a maximum.
c) The global minimum is at the boundary $x = -3$ and the minimum is the other boundary $x = 3$.
d) The Bolzano extremal value theorem.

Problem 9) Algebra (10 points)

a)	$(e^x)^y$	
b)	e^{x+y}	
c)	$\ln(xy)$	
d)	$\frac{\tan(x)}{\sin(x)}$	
e)	$\frac{x^9}{x^3}$	

Choose from the following expressions.

- $1/\cos(x)$
- $\cos(x)/\sin^2(x)$.
- $e^x e^y$
- $\ln(x+y)$
- $e^x + e^y$
- $e^{(xy)}$
- x^6
- $\ln(x) - \ln(y)$
- $e^{(x^y)}$
- $\log(x) + \log(y)$
- x^3
- $\cos(x)$

Solution:
a) e^{xy}
b) $e^x e^y$.
c) $\log(x) + \log(y)$ d) $1/\cos(x)$.
e) x^6.

Problem 10) Linearization (10 points, 5 points each)

a) (5 points) Use linearization to estimate $\ln(e^2 + 0.1)$.

b) (5 points) Use linearization to estimate $\sin(\frac{\pi}{2} - 0.1)$.

Solution:
a) Take the function $f(x) = \ln(x)$ and the point $a = e^2$. We have $f(a) = 2.$ and $f'(a) = 1/a = 1/e^2$. We have therefore $2 + (1/e^2)0.1 = 2.01353$. The actual value is 2.01344.
b) Take the function $f(x) = \sin(x)$ and the point $a = \pi/2$. We have $f(a) = 1$ and $f'(a) = \cos(a) = 0$. The linearized value is 1. The actual value is 0.995. Note that here, because we were at a critical point of the function f, the linearized value was the same than the value at the critical point.