

# INTRODUCTION TO CALCULUS

MATH 1A

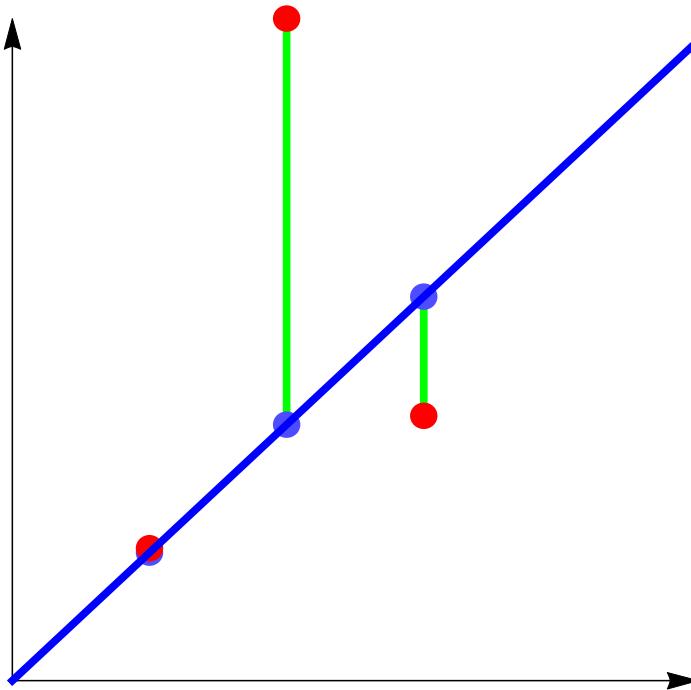
## UNIT 31: WORKSHEET

Linear fitting problems belong to statistics. But one can also see it as an extremization problem. It becomes so a part of economics. Assume we have a couple of data points and we want to find the best line  $y = mx$  through this in the sense that the sum of the squares of the points to the line is minimal. This leads to an extremal problem which is a special case of a **data fitting problem**. It would be more adequate to fit with lines  $y = mx + b$  or more generally with functions but then we have more variables and run into **multivariable calculus** or **linear algebra problems** much outside the scope of this course. But if we have only one parameter, we get a single variable calculus problem.

**Problem 1:** In order to find the best line  $y = mx$  through the points  $(1, 1)$ ,  $(3, 2)$ ,  $(2, 5)$ . we have to minimize the function:

$$f(m) = (m - 1)^2 + (3m - 2)^2 + (2m - 5)^2 .$$

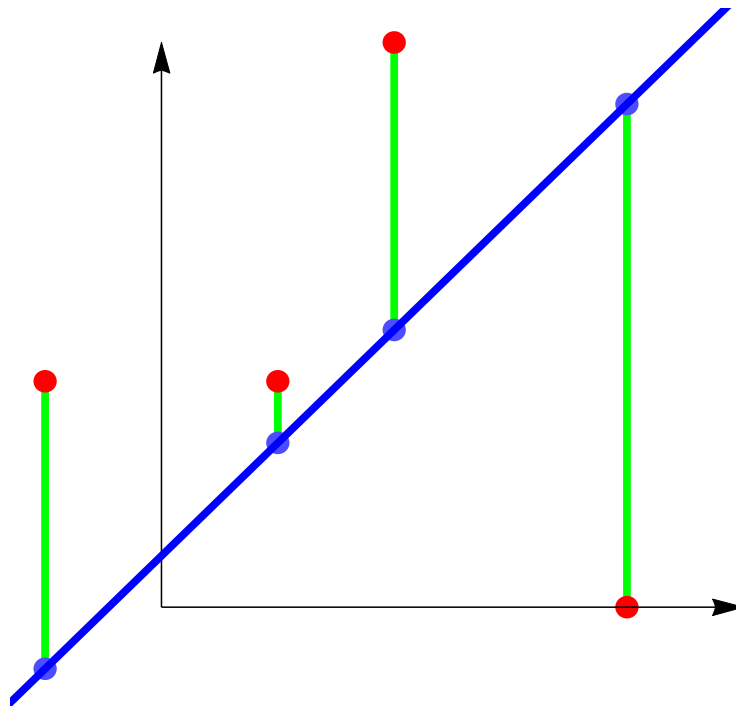
Find the minimum. **Solution.** 17/14



Lets take a different set of data points and look at the problem to fit functions of the form  $y = x + b$ .

**Problem 2:** Find the best line  $y = x + b$  through the points

$(1, 2), (2, 5), (-1, 2), (4, 0), (3, 1)$  .



**Problem 3:** Read the following two pages in "Dominik Heckner and Tobias Kretschmer: Don't worry about Micro, 2008", pages 271-274.

## 9.5 The Connection Between Marginal and Average Costs

This section is slightly more technical than the rest of this chapter. The subject of our analysis at this point is the connection between different cost curves. To be more precise, we investigate how the MC curve cuts through the average cost curves at their respective minima. This is shown in Fig. 9.4.

### Shutdown and Breakeven Points

Before we commence with our analysis, let us link the graph back to some of the discussion above. Looking at Fig. 9.4 you may have noticed that the two quantities for the intersection of MC with AVC and MC with ATC are labelled  $Q_S$  and  $Q_B$ . These are the shutdown and breakeven points, respectively.

The MC curve crosses the AVC and ATC curves at their respective minima.

As we discussed previously, when employing the profit-maximising condition and when AVC is equal to MC, we would be just about indifferent between shutting down or producing in the short run. Secondly, when we set price equal to MC and when at this point MC is also equal to ATC, the firm is just breaking even.

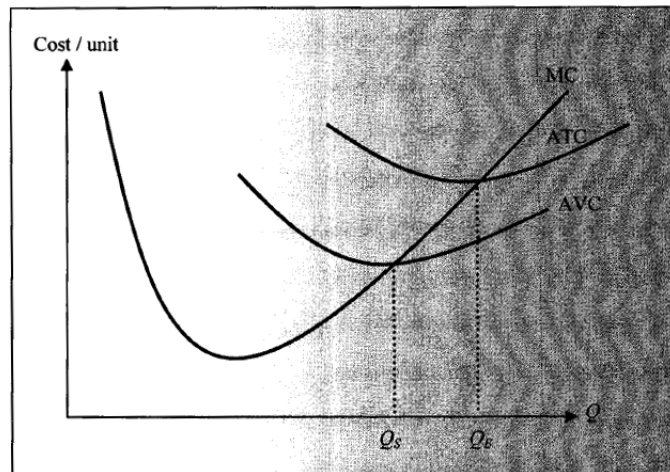


Fig. 9.4. The marginal cost MC curve cuts through average variable cost AVC and average total cost ATC curves at their respective minima. These points are the shutdown and breakeven points, respectively

You take one strawberry after another and place them on a scale that tells you the average weight of all strawberries. The first strawberry that you place in the bucket is very large, while every subsequent one will be smaller (until you reach the smallest one). Because of the literal “weight” of the heavier ones, average weight is larger than marginal weight (i.e. the weight of each strawberry you handle). Average weight still decreases, although less steeply than marginal weight.

Once you reach the smallest strawberry, every subsequent strawberry will be larger, which means that the rate of decrease of the average weight becomes smaller and smaller until eventually it stands still. At this point, the marginal weight is just equal to the average weight.

This logic is an analogy of why MC cuts through the average cost curves at their minimums. The reasoning is identical for both AVC and ATC.

**Mathematical Proof**

Rather than blindly trusting the intuition above, we can also prove our analysis mathematically. Let us perform this proof for the intersection of MC and ATC. Our first step is to compute the derivative of ATC with respect to  $Q$  and set this equal to zero to find the curve's critical point, here the minimum:

$$\frac{dATC}{dQ} = 0 \quad (9.14)$$

In order to make Equation 9.14 usable, let us substitute  $TC/Q$  for ATC. Therefore, we get:

$$\frac{d\left(\frac{TC}{Q}\right)}{dQ} = 0 \quad (9.15)$$

To avoid complicated calculus, let us reformulate the numerator as a product:

$$\frac{d(TC \cdot Q^{-1})}{dQ} = 0 \quad (9.16)$$

Remembering the **product rule**, we differentiate  $TC \cdot Q^{-1}$  with respect to  $Q$  by taking the derivative of the first term and multiplying it by the second, and adding the derivative of the second term and multiplying it by the first. This gives us:

$$\frac{dTC}{dQ} Q^{-1} - TC \cdot Q^{-2} = 0 \quad (9.17)$$

Notice the negative sign between the terms, which is a result of the  $-1$  "brought down" from  $Q^{-1}$  of Equation 9.16 in the process of differentiation. When looking at Equation 9.17, we notice that the very first term is MC and so we can write:

$$MC \cdot Q^{-1} - TC \cdot Q^{-2} = 0 \quad (9.18)$$

As a final step we multiply both sides by  $Q$  and write the second term as a fraction:

$$MC - \frac{TC}{Q} = 0 \quad (9.19)$$

Since, by definition,  $TC/Q$  is equal to ATC, we finalise our equation to become:

$$MC - ATC = 0 \quad (9.20)$$

Now our task of proving that ATC is equal to MC when ATC is at its minimum is easy. Having taken the derivative of ATC in Equation 9.14 to show its minimum, we have worked all the way to Equation 9.20. This last Equation will hold true, i.e. will correspond to a minimum of the ATC curve when we set MC and ATC equal to each other. Hence, when MC is equal to ATC, ATC is at its minimum. The same mathematical steps can be followed to prove the intersection of AVC and MC at the minimum of AVC.