INTRODUCTION TO CALCULUS

MATH 1A

Unit 6: Worksheet

Problem 1: Take the definition of the derivative to find the derivative of the function $f(x) = 2x$.

Solution:
The answer is 2.

Problem 2: Take the derivative of $f(x) = x^4$ by taking limits. We have to simplify

$$\frac{f(x+h) - f(x)}{h} .$$

Start with expanding $(x+h)^4$.

Solution:
We have to foil out $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$. Now, $(x+h)^4 - x^4$ can be divided by h and gives $4x^3 + h(...)$. Setting $h = 0$ gives $4x^3$.

Problem 3: Take the definition of the derivative to look at

$$\frac{f(x+h) - f(x)}{h}$$

in the case $f(x) = |x|$. What happens at $x = 0$?

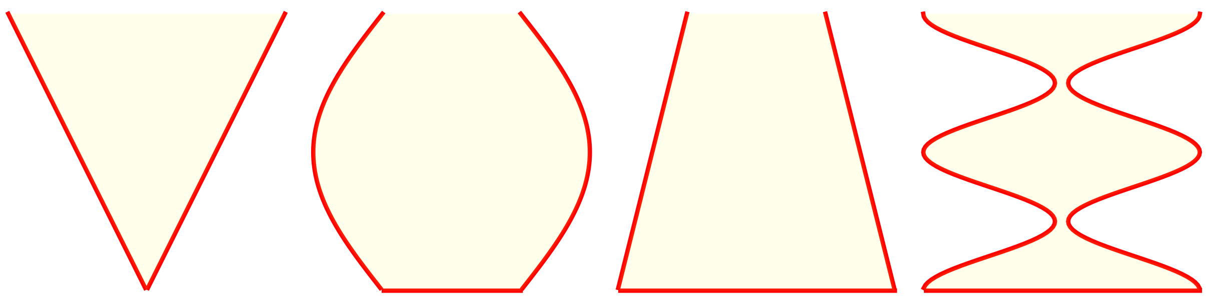
Solution:
The limit is not defined at 0 because the right limit 1 is not the same than the left limit -1.

Problem 4: We start here with the task to find the derivative of $f(x) = \ln(x)$ by taking limits. Simplify $[\ln(x+h) - \ln(x)]$ as much as possible.

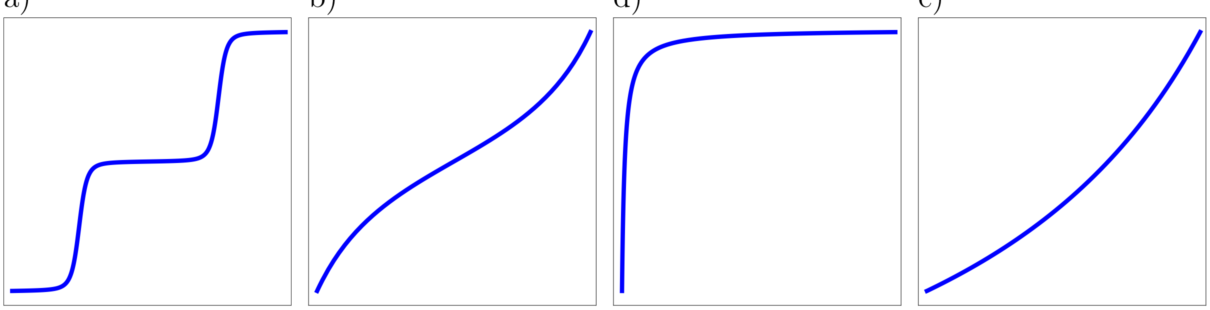
We will continue to work on this next week.

Solution:
We can simplify it to $\ln((x+h)/x) = \ln(1+h/x)$. We will see that $\lim_{h\to 0} \ln(1+h/x)/h$ is $1/x$. This shows that the derivative of ln is $1/x$.

Problem 5: In the famous **bottle calibration problem**, we fill a circular bottle or glass with constant amount of fluid. Plot the height of the fluid in the bottle at time t. Assume the radius of the bottle is $f(z)$ at height z. Can you find a formula for the height $g(t)$ of the water? This is not so easy. But we can find the rate of change $g'(t)$. Assume for example that f is constant, then the rate of change is constant and the height of the water increases linearly like $g(t) = t$. If the bottle gets wider, then the height of the water increases slower. There is definitely a relation between the rate of change of g and f. Before we look at this more closely, lets try to match the following cases of bottles with the graphs of the functions g qualitatively. In each of the bottles, we call g the height of the water level at time t, when filling the bottle with a constant stream of water. Can you match each bottle with the right height function.



a) b) d) c)



1) 2) 3) 4)

Solution:
The key in this problem is to look at $g'(t)$, the rate of change of the height function. Because $[g(t+h)-g(t)]$ times the area πf^2 is a constant times the time difference $h = dt$, we have **bottle calibration formula**

$$g' = \frac{1}{\pi f^2} \; .$$

It relates the derivative function of g with the thickness $f(t)$ of the bottle at height g. It tells that that if the bottle radius f is large, then the water level increase g' is small and if the bottle radius f is small, then the liquid level change g' is large.

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